Commande des Systèmes Hyperboliques décrits par des Equations aux Dérivées Partielles

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par

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Titre de l'habilitation :

Commande des Systèmes Hyperboliques décrits par des Equations aux Dérivées Partielles

Soutenue le 28 avril 2015

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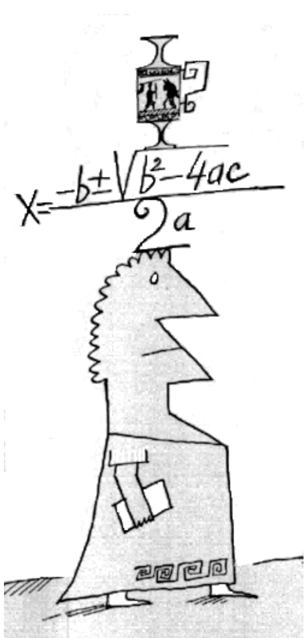
M.	Youssoufi TOURE	Professeur	PRISME, Orléans	

"L'imagination universelle renferme l'intelligence de tous les moyens et le désir de les acquérir"

Extrait des Critiques d'Art,
Charles BAUDELAIRE, écrivain français, 1821-1867.

Pour ma petite Cloé.



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Résumé

Ce travail s'inscrit, d'un point de vue théorique, dans le domaine du contrôle des systèmes décrits par des équations aux dérivées partielles (EDP). L'autre versant de ce travail est l'application concrète à des procédés.

Un grand effort de développement des techniques de modélisation, d'identification et de commande a été réalisé pour les systèmes de dimension finie depuis des années. Ces techniques ont atteint un certain degré de maturité et sont utilisées dans de nombreuses applications. Néanmoins, les développements des technologies de pointes ont entrainé une hausse considérable de la taille des modèles de commande, hausse qui est le reflet dans beaucoup de cas, du passage de la commande d'un vrai système de dimension finie vers un système de dimension infinie.

Depuis quelques décennies, un réel travail de développement des outils en dimension infinie a donc vu le jour. Ces travaux initialement dédiés à des cas plutôt académiques se voient aujourd'hui étendus à des cas pratiques.

Mes travaux se posent à ce niveau : depuis 10 ans je m'intéresse aux problèmes de stabilité et au développement de commandes de systèmes décrits par des EDP hyperboliques. Pour cela, j'utilise des structures mathématiques telles que les semigroupes, des invariants "naturels" comme ceux de Riemann, des structures énergétiques comme les Hamiltoniens, ou par l'extension de résultats existants en dimension finie à la dimension infinie comme pour les LMI (linear matrices inequalities) en LOI (linear operator inequalities).

Tous ces résultats théoriques n'ont d'interêt que s'ils sont appliqués, du moins c'est l'objectif que je souhaite maintenir. A cette fin, tous les résultats ont été développés sur de réels process : les canaux d'irrigation, les voies naviguables, l'extrusion, et d'autres à venir. La problématique de l'eau décrite par les équations de Saint-Venant est certe un exemple central dans mon travail, mais cela est dû simplement au fait que j'ai accès à des bancs d'essais me permettant de valider les approches développées.

L'ensemble de mes travaux a été publié au niveau international, mais aussi diffusé en local lors d'enseignements auprès d'écoles doctorales, lors d'encadrement de masters recherche et de thésards.

Mots–Clés

Commande de systèmes en dimension infinie, contrôle frontière de systèmes décrits par des EDP hyperboliques, théorie de la Perturbation d'opérateurs et de semigroupes, invariants de Riemann, hamiltonien, équations de Saint-Venant, procédés d'extrusion

Abstract

This work is part, from a theoretical point of view, of the control of systems described by partial differential equations (PDE). The other aspect is the application of those results to real process's applications.

Great developments have been done on modelization technics, identification and the control for systems in finite dimension since a long time. Those technics have reached a maturity level, and are applied to numerous applications. Nevertheless, the development of advanced technologies have for consequence to increase the size of the controlled models, rising which is the symbol of the passage from finite dimension system to infinite dimension system.

In recent decades, a real work on the development of infinite dimension tools has emerged. This work originally dedicated to rather academic cases are being extended today to practical cases.

My work has found its place at this level : since 10 years I am interested in stability problems and in the development of controls for systems described by hyperbolic PDE. For this I use mathematical structures such as semigroups, "natural" invariants like the Riemann invariants, energy structures like the Hamiltonian functional, or by the extension of existing results in finite dimension to the infinite dimension such for the LMI (Linear Matrices Inequalities) to LOI (Linear Operator Inequalities).

All these theoretical results have no interest if they are not applied, at least that's the goal I would like to maintain. To this end, all results have been developed on real processes : irrigation channels, navigable waterways, extrusion process, and more to come. The issue of water described by the shallow water equations is a central example in my work, but this is simply because I have access to benchmarks allowing me to validate the developed approaches.

All of my works has been published internationally, but also broadcasted on lessons from doctoral schools, in training of masters students and PhD students.

Key–Words

Infinite dimension systems control, boundary control of systems described by hyperbolic PDE, perturbation theory of operators and semigroups, Shallow water equation, extrusion process

Remerciements

Je tiens d'abord à exprimer toute ma reconnaissance à mon directeur de thèse, Youssoufi TOURÉ, pour m'avoir initiée à la recherche, pour ses qualités humaines et pour ses qualités scientifiques, qui m'ont permis de réaliser ma carrière de maître de conférences. Qu'il trouve dans ce travail toute ma gratitude pour son amitié tout le long de ces années.

Je souhaite également remercier très chaleureusement Claire VALENTIN, Mickael RODRIGUES, Audrey JUBAN et Emilie GAGNIERE pour leur amitié, pour leur soutien moral et tout simplement d'avoir été présents. Merci également pour ses longues discussions, Mick, et ses 7 ans de collaboration. Une mention particulière pour Fayez S. AHMED pour son aide précieuse.

Ma gratitude va également à messieurs Nicolas PETIT, Miroslav KRSTIC et Emmanuel TRELAT qui me font l'honneur d'être rapporteurs de mon manuscrit d'habilitation. Veuillez trouver en ces quelques lignes ma profonde estime à votre égard.

Merci également à Mireille HUBERT-BATTON pour nos échanges passés et à venir, ainsi que de me faire le plaisir de participer à mon jury.

Je suis aussi très reconnaissante à Luc DUGARD, pour avoir partagé son bureau pendant quelques mois avec moi, sa patience, son écoute et d'avoir accepté d'être membre de mon jury.

Merci à tous les membres du GIPSA-lab qui m'ont accueillie l'an passé, en espérant que nos projets de recherche n'en sont qu'à leur prémisses.

Je tiens à remercier tout spécialement Nadia CHAPEL notre assistante de direction pour son dévouement au laboratoire, sa grande efficacité, pour son entrain et sa bonne humeur... s'il te plaît ne changes pas !

Je remercie également le professeur Bernhard MASCHKE pour m'avoir accueillie dans l'équipe DYCOP, et de m'avoir laissée une grande autonomie de travail.

Merci à tous les membres du LAGEP et de Polytech'Lyon (que ce soit sur Roanne ou Lyon) que j'ai cotoyés tout au long de ces années, merci pour tout ce que nous avons partagé. Une mention particulière pour Naima DEBIT, Sophie CAVASSILA et Christian CACHARD dont le soutien a été pour moi très important.

Je dédie ce travail à mes parents, mon frère Israël, sa femme Cristelle et à mes amours de neveux Ilan et Joris.

Je dédie mes recherches aux miens : mon ange Cloé, ma princesse Lucia et leur papa Didier. Je vous remercie pour votre soutien, votre amour et votre confiance.

Villeurbanne, 28 avril 2015 — Valérie DOS SANTOS MARTINS

Introduction

"La recherche dans le domaine du contrôle des systèmes à paramètres distribués, décrits par des Equations aux Dérivées Partielles (EDP) est un champ d'investigation pérenne depuis plusieurs décennies.

Pendant longtemps, les résultats théoriques les plus avancés ont concerné des cas plutôt académiques, mais avec un aspect générique non négligeable.

Ces dernières années, beaucoup de classes d'EDP étudiées concernent des problèmes décrivant une finesse ou une exigence de fonctionnement de systèmes réels.

Parallèlement, le développement de la théorie du contrôle en dimension infinie, i.e. dans des espaces fonctionnels de Banach (ou Hilbert), permet d'appréhender beaucoup plus de problèmes en gardant cette description relativement fine que constitue la description par des EDP.

Dans le domaine de l'automatique, les problèmes de stabilisation, de poursuite de trajectoire, de régulation, etc.. se posent formellement de la même façon quelle que soit la description adoptée dans un modèle. L'analyse du système, à travers ce modèle, introduit une distinction dans les différentes approches pour traiter les problèmes précisés. Ainsi, il est possible de garder les concepts formels simples pour traiter des problèmes relativement complexes."

Mon introduction de thèse s'amorçait ainsi, et aujourd'hui cela est toujours d'actualité tant ce domaine est vaste et englobe différentes approches et différents point de vue.

Mes activités de Recherche ont débuté en 2000, lors de mon stage de DEA au CEA sur les milieux granulaires. A l'époque, j'ai rencontré Youssoufi Touré, alias Chef, professeur en Automatique (LVR, Bourges) venu au laboratoire de mathématiques le MAPMO de l'université d'Orléans pour "diversifier" la culture scientifique des "théoriciens" en herbe que nous étions, en leur montrant que les mathématiques sont un outil qui s'utilise dans beaucoup de disciplines, entre autre en automatique.

De cette rencontre est née une thèse sur les EDP, thèse que j'ai soutenue en 2004. Dix ans après, le thème des EDP est resté mais les techniques et approches se sont diversifiées.

Après ma thèse j'ai obtenu un demi-poste d'ATER, transformé en temps plein, mais j'ai continué à travailler sur les semi-groupes et rencontré lors de conférences des collègues avec lesquels j'ai eu la chance de travailler par la suite. Georges Bastin fut l'un d'entre eux, et nous commençâmes une collaboration qui se concrétisa sous forme de post-doctorat, pour lequel Georges fût mon promoteur. J'abordais alors une nouvelle approche sur les EDP via les invariants de Riemann. Durant cette année de recherche, il m'a été donné l'opportunité de travailler avec des collègues de haut niveau comme Jean-Michel Coron (Laboratoire Jacques-Louis Lions, UPMC/Paris-Diderot/CNRS), Brigitte d'Andréa-Novel (Mines ParisTech), et d'autres moins connus à l'époque qui depuis ont pris du grade comme Christophe Prieur (GIPSA-lab, Grenoble).

En 2006, j'intègre le LAGEP, UMR CNRS 5007, à l'université Claude Bernard Lyon1, dans l'équipe Systèmes Dynamiques, actuellement DYCOP (Dynamique et Commande des Procédés). De nouveau, j'aborde le problème des EDP sous un nouveau jour, via une approche Hamiltonnienne, des invariants de Casimirs, des structures de Dirac. Plusieurs projets ont vu le jour, dont une thèse en collaboration avec certains collègues de l'équipe et une thèse en collaboration entre DYCOP et Denis Dochain, professeur au CESAME (Belgique). Dans le cadre d'un projet ANR, nous avons débuté une collaboration avec Thomas Helie, CR HDR à l'IRCAM de Paris, sur des problèmes d'acoustique.

En parallèle, je rencontre Mickael Rodrigues, MCU LAGEP arrivé en septembre 2006 également, qui travaille sur les Multi-Modèles. Ce thème m'intéresse fortement, puisque c'est un point théorique resté en suspens lors de ma thèse, problème soulevé par les expérimentations que j'avais effectué quelques temps auparavant. Nous travaillons ensemble depuis 7 ans pour généraliser l'approche Multi-Modèles très usitée en dimension finie au cas de la dimension infinie. De nombreuses publications sont liées à cette thématique bien que seule l'opportunité d'encadrer des masters nous ait été offerte.

L'an passé se déroule au laboratoire GIPSA-lab, Grenoble Images Parole Signal Automatique, une unité mixte du CNRS et de l'université de Grenoble, en délégation CNRS pendant 8 mois. Trois nouvelles collaborations ont vu le jour, toujours avec une empreinte EDP et une nouvelle fois sous de nouveaux aspects.

Ce manuscrit se divise en deux parties : la première fait la synthèse de mes activités, sous forme résumé dans le chapitre 1, puis plus explicite pour l'enseignement dans le chapitre 2, et pour la recherche dans le chapitre 3.
La seconde partie est dédiée à mes travaux de recherches, et si tous les thèmes ne sont pas explicités, je reviens sur les parties les plus représentatives dans les chapitres qui la constituent.

Je termine enfin par la présentation des différentes collaborations et projets en cours et une reflexion sur mon projet de recherche en conclusion. En annexes, se trouvent mes principales publications.

Synthèse des activités
Synthesis of the activities

CHAPITRE 1

Curriculum Vitae et résumé d'activités

"*Vous le savez tout comme moi : ce qui reste d'une existence, ce sont ces moments absents de tout curriculum vitae et qui vivent de leur vie propre ; ces percées de présence sous l'enveloppe factice des biographies. Une odeur un appel un regard et voilà les malles, les valises, les ballots solidement arrimés dans les soutes qui se mettent en mouvement, s'arrachent aux courroies et aux cordages et vont faire chavirer le navire de notre raison quotidienne !*
Non qu'à ces moments-là nous devenions fous. Loin de là. Un instant, à l'enfermement, à l'odeur confinée du fond de navire a succédé le vent du large. L'illimité pour lequel nous sommes nés se révèle."

Les sept nuits de la reine, Christiane Singer, Ecrivain française (1943-2007).

Sommaire

1.1 Curriculum Vitae

Valérie DOS SANTOS MARTINS,
née le 08 janvier 1976.
Célibataire, 1 enfant.

Maître de Conférences en Automatique au Laboratoire d'Automatique et de Génie des Procédés (LAGEP, UMR CNRS 5007), et de l'Institut des Sciences et Techniques de l'Ingénieur de LYON (ISTIL) anciennement, et nouvellement école Polytech'Lyon, école d'ingénieur interne de l'Université Claude Bernard Lyon1.

Docteur en Mathématiques Appliquées, soutenue au Laboratoire de Vision et Robotique (LVR-UPRES EA 2078), Université d'Orléans.

LAGEP, UMR CNRS 5007	**Polytech'Lyon**	
Laboratoire d'Automatique et de Génie des Procédés Université Claude Bernard Lyon 1 bât 308G ESCPE-Lyon, 2ème étage 43 bd du 11 Novembre 1918 69622 Villeurbanne Cedex Tel : +33 (0) 4 72 43 18 52 Fax : +33 (0) 4 72 43 16 82	Site de la Doua Université Claude Bernard Lyon1 Domaine Scientifique de la Doua 15 boulevard André Latarjet 69622 Villeurbanne Cedex Tel : +33 (0) 4 72 43 27 12 Fax : +33 (0) 4 72 43 12 25	Site de Roanne Université Claude Bernard Lyon1 Technopole Diderot, 1, rue Charbillot 42300 Roanne, France Tel : +33 (0)4 77 23 63 90 Fax : +33 (0)4 77 23 63 99
Email : dossantos "at" lagep.univ-lyon1.fr http ://www.lagep.cpe.fr/wwwlagep7/annuaire/ ?id=403		

1.2 Positions Universitaires

2006	**Maître de Conférences**, 61^{me} section, LAGEP, UMR CNRS 5007, Polytech'Lyon, filière principale Systèmes Industriel et Robotique (filières secondaires : mécanique et informatique).
2005-2006	**Post Doctorat** Département d'ingénierie mathématique CESAME/ INMA Université catholique de Louvain (Belgique).
2004-2005	**ATER** 26ème section, département de Mathématique, Université d'Orléans.
2001-2004	**Vacataire**, département GMP, GEA, OGP, formation continue SEFCO, à l'IUT de Bourges.

1.3 Diplômes et formations

2005-2006	**Post Doctorat** Département d'ingénierie mathématique CESAME/ INMA, Louvain la Neuve, Université catholique de Louvain (Belgique), sous la direction du Pr Georges BASTIN
2005	**Qualifiée** en sections **CNU 26** "Mathématiques Appliquées" et **CNU 61** "Génie Informatique, Automatique, Traitement du Signal".
2004-2005	**ATER** attachée au Laboratoire de Mathématiques, Applications et Physique Mathématique d'Orléans (MAPMO), Université d'Orléans.
2001-2004	**Thèse de Doctorat** de Mathématiques appliquées, Université d'Orléans, "Contrôle Frontière par Modèle Interne de Systèmes Hyperboliques : Application à la Régulation de Canaux d'Irrigation" sous la direction du Pr Youssoufi TOURÉ, Laboratoire de Vision et Robotique (LVR, Bourges)
2000-2001	**DEA** "Analyse Mathématique et Applications", Université d'Orléans, Mention assez bien, Mémoire sur les milieux granulaires, CEA de Bruyères le Châtel, Paris

1.4 Responsabilités collectives

1.4.1 Laboratoire

• **Juin 2007** : organisation de la journée Jeunes Chercheurs JJC2007 en collaboration avec Mickael Rodrigues (50%-50%)
- Organisation administrative : intendance, salle, déjeuner...
- Organisation scientifique : collecte des résumés, programme de la journée, invitation des présidents de sessions, mise en place d'un livret des résumés, mise en ligne des présentations...

• **Novembre 2007** : organisation du colloque STIC Environnement. Le LAGEP a co-organisé avec le CEMAGREF et GIPSA-Lab, le 5ème colloque STIC & Environnement qui s'est déroulé à Lyon.
- Membre du comité d'organisation : intendance, salle, déjeuner... (30%)
- Membre du comité de programme : lancer l'appel à contribution, invitation de personnalités, programme de la journée ... (50%)
- Membre du comité scientifique : distribution et attribution des contributions scientifiques, relecture,... (50%)
- Présidente de session STIC&Env07 intitulée " 2 : Méthodes et Outils de Modélisation Mathématique "
- Editeur du numéro spécial de la revue E-sta, ISSN : 1954-3522, Volume 5, N^{o}2 - 1er trimestre 2008 - Spécial STIC&Environnement'07 sous la tutelle du Pr. Touré (75%)
- Membre du comité de lecture de la revue E-sta

• **Depuis mai 2009** : **gestionnaire** de la page web de l'équipe Systèmes Dynamiques, LAGEP à 100%.

• **Depuis octobre 2014** : **gestionnaire** du site web du laboratoire LAGEP à 100%.

1.4.2 Ecole ISTIL/Polytech'Lyon

Mes activités au sein de l'école sont diverses. Notons que je suis attachée principalement à l'antenne de Roanne. De ce fait et pour garder un contact permanent avec le site de Lyon, je me suis investie dans diverses tâches administratives.

• **2006-2008** et **2009-2011** : membre nommé à la commission pédagogique de l'école par le directeur Joseph Lieto.

• **Mars 2007** : organisation de l'inauguration de la plate-forme technologique de l'antenne ISTIL à Roanne en présence de la CCI de Roanne, du Conseil Régional Rhône-Alpes.

• **2011-2015** : membre élu au CEVE (Conseil des études et de la vie étudiante) et au conseil de Filière SI.

• **2008-2013** : responsable des modules Automatique, Plan d'expérience en 4A, et Modélisation par Bond Graph et TP Plateforme et logistique en 5A.

• **2009-2015** : responsable du module Méthode Numérique de Base en 3A.

1.4.3 Université

• **Depuis 2008** : membre élu au Conseil Consultatif (CoC) de la section 61.

• **Depuis 2008** : participation au Comité de Sélections (CoS) pour les recrutements de postes MCU (2 en MCU 61, 1 en MCU 61/63) et ATER (6).

• **Depuis 2012** : membre enseignant du GTVE (Groupe de Travail Vie Etudiante) élue par le CEVU (réunion mensuelle).

1.4.4 Cadre Régional

• **2006-2012** membre élue au Conseil d'administration de l'Agence du Développement Economique de la Loire, dans le collège de la Recherche et de l'Enseignement Supérieur (réunion trimestrielle).

1.4.5 National

• **2007-2012** : membre comité scientifique STIC depuis 2007, présidente de session, membre du jury pour le prix du meilleur papier jeunes chercheurs.

• **2012** : évaluateur pour le Prix des meilleures thèses du GdR MACS.

1.4.6 International

• **Depuis 2011** : membre des Comité Editeurs du journal international "Computer Engineering and Technology" (IJCET), publié par "American V-King Scientific Publishing, LTD".

• **Depuis 2011** : membres du Comité Technique de l'IFAC (International Federation of Automatic Control) T.C. 2.6 "Distributed parameter Systems ".

• **2011** : membre du Comité IPC de la conférence IFAC-CPDE, Paris (Control of systems Modeled by Partial Differential Equations).

• **Depuis 2014** : éditeur technique associé au comité technique international (IPC Technical Associated Editor) du World Congress of the IFAC (International Federation of Automatic Control) à Cape Town (Afrique du Sud).

• **2014-2017** : membre du Comité d'organisation scientifique et du comité national du World Congress of the IFAC (International Federation of Automatic Control) 10-14 juillet 2017 à Toulouse.

• **2013** : organisation de sessions invitées dans des conférences internationales (dernière en date : IFAC-CDPE 2013)

• **Depuis 2006** : présidente et co-présidente de session dans des conférences internationales et nationales (JJC, STIC & Environnement, CIFA, IFAC SSC, IFAC CPDE...).

1.5 Activités de recherche et formation de jeunes chercheurs

Mon activité de recherche concerne les problèmes de commande des systèmes hyperboliques décrits par des équations aux dérivées partielles. Plus précisement, je me suis intéresssée aux problèmes de stabilité de tels systèmes d'une part, et au développement de commande par tracking ou dans un objectif de régulation d'autre part. Trois axes se dégagent de mon travail :
- les systèmes hamiltoniens à port : invariants, lien des structures Hamiltoniennes et Riemanniennes...
- la stabilité et le contrôle de procédés à frontière variable,
- stabilité par inégalités linéaires d'opérateurs.

1.5.1 Projets de recherche

Pour plus de détails cf. section 3.4.
• 2006 et 2008-2009 : PAI TOURNESOL (Belgique), programme de coopération franco-belge
<u>Rôle</u> : participant
<u>Partenaires</u> : LAGEP UMR CNRS 5007, CESAME UCL Belgique.
• 2006-2009 : ANR PARADE (ANR Technologie Logicielle Numérique, Parallel numerical Algorithms for Real time simulation of Algebraic Differential Equations systems).

Rôle : participation au comité de recrutement de l'étudiant en thèse pour le LAGEP, participant
Partenaires : "**CDCSP**", Institut Camille Jordan (ICJ) - UMR5208 Un. Lyon 1-ECL-INSA-CNRS, "**LAGEP**" - UMR 5007 Univ. Lyon 1-CNRS, **LIMOS**, Université d'Auvergne Clermont I et Université Blaise Pascal (UBP) Clermont II (UMR 6158, CNRS), **METALAU** INRIA, institut national de recherche en informatique et en automatique, **IMAGINE** Société de Services en Ingéniérie Systèmes (Roanne) et **Siemens** VDO Automotive SAS.
• 2011-2015 : Projet ANR Blanc Approche Hamiltonienne pour l'analyse et la commande des systèmes multiphysiques à paramètres distribués (HAMECMOPSYS).
Rôle : participation au comité de recrutement de l'étudiant en thèse pour le LAGEP, participant
Partenaires : FEMTO-ST/AS2M, UMR CNRS, Besançon, LAGEP, UMR CNRS 5007 , Institut Supérieur de l'Aéronautique et de l'Espace (ISAE), Toulouse , Université de Nancy, Institut Elie Cartan.

1.5.2 Collaborations internationales

• **G. Bastin**, Université Catholique de Louvain, Louvain-la-Neuve, Belgique.
Collaboration sur l'approche par Invariants de Riemann sur des systèmes hyperboliques décrits par des équations aux dérivées partielles, avec une application aux voies navigables.
2 journaux [Automatica 2007, Journal of Tomography & Statistics CAO-2007]
2 congrès [Nolcos 2007, 13th IFAC Workshop on Control Applications of Optimisation CAO-2006]

• **D. Dochain**, Université Catholique de Louvain, Louvain-la-Neuve, Belgique.
Collaboration sur la génèse de fonctions pour la commande de procédés thermodynamiques irreversibles par des méthodes basées sur l'entropie des systèmes et les variables Hamiltoniennes.
1 journal [IEEE TAC 2009]
1 congrès [IEEE CSS, European Control Conference ECC-2009]

• **M. Krstic**, Université de San Diego, Californie, USA.
Mise en place d'un projet sur la commande de procédés de grandes tailles, type forage/ éoliennes... Projet initié et pour lequel le professeur Krstic viendra dans le cadre des mois invités en mai 2015.

1.5.3 Diffusion scientifique

• **2011 : JD-JN MACS - Ecole des JDMACS** - Course on Control of Distributed Parameter Systems, avec Emmanuel Trélat, Yann Le Gorrec, Laurent Lefèvre.
Intitulé du cours : Some results on the stability of nonlinear hyperbolic systems.

• **2010-2011-2012 : Cours pour l'école doctorale EEA, UCBL/INSA/ECL Lyon**
Intitulé du cours : Commande des systèmes régis par des EDP dans les cas linéaire et non linéaire - Coordonnées et invariants de Riemann

• **2014 : Cours de M2R MiSCIT et pour la formation doctorale en Automatique EEATS, INP de Grenoble**

Intitulé du cours : Hyperbolic systems, from the statement to the stability

1.5.4 Travaux d'expertise

Je suis référente pour les revues suivantes depuis 2006

- AUTOMATICA,
- IEEE TAC (Transactions on Automatic Control),
- IEEE TCST (Transaction on Control Systems technology),
- SICON (SIAM Journal on Control and Optimization),
- Systems & Control Letters,
- IMA (Journal of Mathematical Control and Information),
- MCSS (Mathematics of Control, Signals, and Systems),
- JCET (Journal of Control Engineering and Technology),
- JESA (Journal Européen des Systèmes Automatisés)/ E-sta,

et pour des Congrès comme ECC (European Control Conference), CDC (Conference on Decision and Control), IFAC World Congress, CDPS (Control of Distributed Parameter Systems), IEEE ACC (American Control Conference), NOLCOS (Nonlinear control systems), IEEE CSS (Control Systems Society Conference), CDPE (Conference on Differential Partial Equations) ...

1.5.5 Appartenance à des sociétés savantes

Je suis membre de sociétés savantes :

- SMAI (Société de Mathématiques Appliquées et Industrielles)
- SMF (Société Mathématique de France)
- GDR MACS (Groupe de Recherche : Modélisation, Analyse et Conduite des Systèmes Dynamiques)

Je suis membre de groupes de travail :

- GT Contrôle du Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie (Paris 6)
- GT EDP, Théorie et applications des la commande des systèmes à paramètres répartis

1.5.6 Encadrement de jeunes chercheurs

• Co-encadrement d'une thèse soutenue :

▶ M. L. DIAGNE, *Etude des systèmes de lois de conservation à interfaces mobiles : application à un procédé d'extrusion*, soutenue le 26 juin 2013, allocation ministère MESR, co-encadrement à 30% avec B. Maschke et F. Couenne, 2 journaux dont 1 commun, 4 congrès dont 2 communs.
Cadre : ANR HAMECMOPSYS

• Collaboration à une thèse soutenue :

▶ Audrey Favache, *Thermodynamique et commande des procédés*, sous la direction du Professeur Denis DOCHAIN, à hauteur de 10%, en fin de thèse, 2 publications communes (1 journal, 1 congrès).
Cadre : projets PAI Tournesol.

• Co-encadrement de 5 masters dont un en cours (détails reportés dans la section 3)

▶ 2007 : **Yves Hagedorn**, à 100%
Sujet : " Etude de stabilité non linéaire d'un écoulement uni-directionnel par fonction de Lyapunov : une approche Hamiltonienne".

▶ 2008 : **Samir Chabou**, co-encadré avec Mickael Rodrigues à 50%, MCU LAGEP.
Sujet : " Commande et l'analyse des systèmes hydrauliques par une approche multi-modèles en dimension finie et/ou infinie ".

▶ 2009 : **Mamadou Diagne**, co-encadré avec Mickael Rodrigues à 50%, MCU LAGEP.
Sujet : " Commande et l'analyse des systèmes hydrauliques par une approche multi-modèles en dimension finie et/ou infinie " .
2 publications : 1 congrès, **1 journal**

▶ 2013 : **Yongxin Wu**, co-encadré avec Mickael Rodrigues à 50%, MCU LAGEP.
Sujet : "Commande et analyse des systèmes hydrauliques par une approche Multi-Modèles en dimension infinie" .
3 publications : 1 congrès [ECC 2013], **1 journal** [IEEE TCST 2013], 1 séminaire invité [SAR-2012]

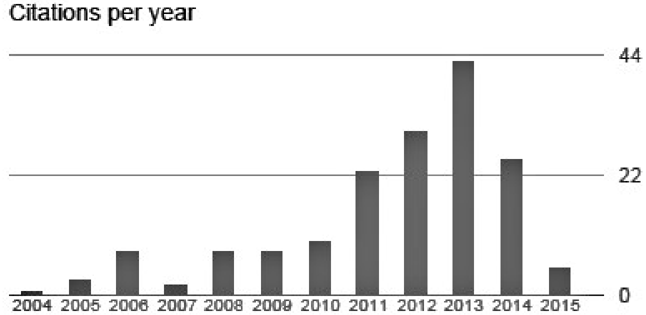
▶ 2015 : **Alexandre de Terrand-Jeanne**, à 100%.
Sujet : "Control methodology for large scale electro-mechanical systems" .
En cours...

1.5.7 Publications scientifiques

11 articles de journaux avec comité de lecture
22 congrès internationaux avec comité de lecture dont une en tant qu'invité, trois en sessions invitées
3 congrès internationaux sans comité de lecture
5 communications nationales dont 1 invitée
22 séminaires dont 5 invitée

Liste complète : `https://sites.google.com/site/vdossantosmartins/home/vdsm_publications` et reportée à la section 3

Citations : 155 dont 131 depuis 2009 **H-index :** 6
`http://scholar.google.fr/citations?hl=fr&user=88zQ7fAAAAAJ`



CHAPITRE 2

Synthèse des Activités d'Enseignement

"*L'enseignement devrait être ainsi : celui qui le reçoit le recueille comme un don inestimable mais jamais comme une contrainte pénible.*"

Albert Einstein, physicien allemand, 1879-1955.

Sommaire

Mes activités d'enseignement ont débuté lors de ma thèse en tant que vacataire pour l'IUT de Bourges dans les départements de Génie Mécanique et Productique (GMP) et Gestion des Entreprises et des Administrations (GEA), principalement pour de l'enseignement de bureautique. Par la suite je suis également intervenue pour la formation continue (SEFCO) et en Organisation et Génie de la Production (OGP) pour des cours de mathématiques.

Mon expérience d'enseignant s'est étoffée lors de mon 1/2 poste d'ATER (transformé en poste temps plein) à l'université d'Orléans, ainsi que lors de mon post-doctorat à l'université de Louvain la Neuve. Dans les deux cas, j'ai enseigné les mathématiques à un public varié, puisque allant des licences de physique, de biologie, de mathématique à des élèves ingénieurs.

Enfin en 2006, j'ai intégré l'Institut Sciences et Techniques de l'Ingénieur de Lyon, école d'ingénieur de l'université Claude Bernard Lyon1, devenue depuis Polytech'Lyon (insertion dans le réseau Polytech en janvier 2012) pour laquelle mon activité d'enseignement s'est principalement concentrée sur la filière Systèmes Industriels et Robotique (SIR). Cette filière est délocalisée sur la demande de l'agglomération Roannaise et la Chambre du Commerce qui souhaitent intégrer les ingénieurs à son tissu industriel fort.

Ce chapitre présente de manière assez détaillée l'ensemble des modules dans lesquels je suis intervenue, ainsi que les volumes horaires associés. Les responsabilités administratives liées sont également présentées.

2.1 Résumé des activités d'enseignement

• **Depuis 2006** :

▶ Cours présentiel à Polytech'Lyon, anciennement l'Institut des Sciences et Techniques de l'Ingénieur de LYON (ISTIL) ($\simeq$240 heures éq. TD par an) :

- Bond Graph, 5$^{\text{ème}}$ année filière SI Roanne, **(37.5 HeqTD)**
- TP orientés systèmes industriels, 5$^{\text{ème}}$ année filière SI Roanne, **(45 HeqTD)**
- Systèmes discrets, 4$^{\text{ème}}$ année filière SI Roanne, **(52.5 HeqTD)**
- Plans d'expériences, 4$^{\text{ème}}$ année filière SI Roanne, **(40 HeqTD)**
- Méthodes Numérique de Base, 3$^{\text{ème}}$ année filières SI Roanne/ Méca/Info Lyon, **(40 HeqTD)**
- Automatique continu, 3$^{\text{ème}}$ année filières SI Roanne et Méca Lyon, **(28 HeqTD)**.

▶ *Enseignement à caractère scientifique* :

- Cours pour le **JN-JD MACS (3h de cours)**
- Tutoriales pour l'**école doctorale EEA**, UCBL/INSA/ECL de Lyon **(3 HeqTD)**
- Cours pour le **M2R MiSCIT/école doctorale EEATS**, INP de Grenoble **(15h de cours)**

▶ *Autres enseignements* :

- Encadrement et suivi de stages entreprises, 4$^{\text{ème}}$ et 5$^{\text{ème}}$ années filière SI Roanne,
- Encadrement de projet de fin d'étude (PFE)
- Encadrement de projet pour l'AUP (année universitaire préparatoire) **(20 HeqTD)**,
- Compléments Mathématiques, 3$^{\text{ème}}$ année, filière SI Roanne **(15 HeqTD)**

• **Avant 2006** : Mathématique de premier cycle

- Université Catholique de Louvain 2005-2006, Belgique, **(70 HeqTD)**

– Service complet ATER MAPMO, Université d'Orléans 2004-2005
– IUT de Bourges 2002-2003 **(35 HeqTD)**
– Service de Formation continue de l'Université d'Orléans 2002, **(45 HeqTD)**

▶ *Autres enseignements* : Bureautique de premier cycle
– IUT de Bourges 2001-2004 **(97 HeqTD)**

2.2 Enseignement durant la thèse (177 HeqTD)

2.2.1 Tableau récapitulatif des enseignements de 2001 à 2004, au sein de l'IUT de Bourges

• **2002 -2003 Organisation & Génie de la Production (OGP)**

Intitulé de la composante de Module	Niveau	Cours (h)	TD (h)	TP (h)
Mathématique 2002-2003	IUT 1	11.5		26.5
Total :		11.5		26.5

Equivalent TD : 35 h

• **2002 SErvice de Formation Continue de l'Université d'Orléans (SEFCO)**

Intitulé de la composante de Module	Niveau	Cours (h)	TD (h)	TP (h)
Mathématique Mise à Niveau	Bac 1	30		
Total :		30		

Equivalent TD : 45 h

• **2001 -2003 Département Génie Mécanique & Productique (GMP)**

Intitulé de la composante de Module	Niveau	Cours (h)	TD (h)	TP (h)
Bureautique 2001-2002	IUT 1		15	12.5
Bureautique 2002-2003	IUT 1			25
Total :			15	37.5

Equivalent TD : 40 h

• **2003 -2004 Gestion des Entreprises & Administration (GEA)**

Intitulé de la composante de Module	Niveau	Cours (h)	TD (h)	TP (h)
Bureautique 2003-2004	IUT 1		57	
Total :			57	

Equivalent TD : 57 h

Total **: 57 + 40 + 45 + 35 = 177 heures équivalents TD**

Tableau 2.1 : *Résumé des enseignements effectués en tant que vacataire*

2.2.2 Description détaillée des enseignements de 2001 à 2004, au sein de l'IUT de Bourges

• Enseignements SEFCO : Mathématiques
En juin 2002, la SEFCO m'a proposé de donner des cours et TD de mise à niveau en mathématique pour des salariés de GIAT Industries, dans le cadre de la formation continue. J'ai entièrement conçu les cours, les exercices et l'examen, ainsi que sa correction.

Grandes lignes du contenu :
- rappels de base sur les fonctions, fonctions trigonométriques, les nombres complexes,
- dérivées,
- fonction exponentielle, fonctions puissances.

• Enseignements OGP : Mathématiques
Lors de ma deuxième année de thèse (2002-2003), une partie des cours et TD de mathématique des premières années du département OGP m'a été confiée.

Grandes lignes du contenu : (Algèbre)
- les nombres complexes,
- les polynômes,
- les fractions rationnelles.

• Enseignements GMP : Bureautique
Les Travaux Pratiques de Bureautique enseignés en GMP ont pour objectif de familiariser les étudiants avec Word et Power Point pour la rédaction de leur Mémoire de stage et la préparation de leur présentation (2001-2003).
Le but est de leur montrer l'étendue des capacités de Word en tant que logiciel de rédaction, en leur apprenant à s'en servir et en leur donnant les bases de la rédaction d'un rapport écrit, que ce soit sous le point de vue de la forme comme du fond. Il en va de même pour Power Point, l'objectif est de leur montrer la nécessité d'avoir un support écrit bien construit et clair pour réaliser une bonne présentation orale. Les travaux pratiques sur Excel ont servi à leur apprendre à résoudre divers problèmes qu'ils pouvaient rencontrer à l'aide des macros d'Excel.

• Enseignement GEA : Bureautique
Les Travaux Dirigés de Bureautique enseignés en GEA ne sont pas dans la même optique que précédemment. En effet, Word, Excel et Access sont des outils de travail que les étudiants de GEA devront utiliser couramment dans leur vie professionnelle (2003-2004). L'enseignement Excel que je leur ai donné (deux groupes sur cinq) est réparti sur quinze TD, qui mettent en scène la gestion informatique de factures et de divers événements. L'utilisation de macros est abordée.
Dans ces trois derniers cas, j'ai remplacé des enseignants partis en congé maladie ou maternité. Mon travail a donc été le suivant :
- prendre connaissance des TD-TP,
- préparer les cours, les TD-TP,
- élaborer et corriger les sujets d'examens partiellement ou entièrement.

2.3 Enseignement durant l'ATER (192 HeqTD)

2.3.1 Tableau récapitulatif des enseignements de 2004 à 2005, au sein de l'Université d'Orléans

• **2004 -2005 Licence Sciences et Technologies (UFR Orléans)**

Intitulé de la composante de Module	Niveau	Cours (h)	TD (h)	TP (h)
Mathématiques	Lic 2		44	
Mathématiques	Lic 1		148	
Mathématiques	Lic 2	8		
Total :		8	192	

Total : **204 heures équivalents TD**

Tableau 2.2 : *Résumé des enseignements effectués en tant que ATER*

2.3.2 Description détaillée des enseignements de 2004 à 2005, au sein de l'Université d'Orléans

• **Enseignements Licence de Physique, deuxième année : Mathématiques**
Les premiers enseignements qui m'ont été confiés sont en Licence deuxième année de Physique au sein de l'Université d'Orléans, il s'agit de cours de Mathématiques portant sur :

– l'analyse pour le premier semestre,
– l'algèbre pour le second semestre.

Un groupe de TD était sous ma responsabilité, pour chaque semestre (2 * 22 heures TD).

• **Enseignements Licence de Biologie, deuxième année : Mathématiques**
Il s'agit d'un cours d'initiation aux statistiques (moyenne, espérance, test de χ^2) pour donner à des biologistes quelques outils de comparaison et d'étude de leur donnée.

• **Enseignements Licence de Mathématiques, première année : Mathématiques**
Au second semestre de l'année universitaire 2004-2005, mon enseignement a porté sur le module d'Algèbre et d'Analyse 1, deux groupes de TD m'ont été confiés (2 * 74 heures TD). Le programme est le suivant :

– algèbre linéaire de dimension finie,
– systèmes linéaires, diagonalisation,
– continuité,
– théorème de Rolle, accroissements finis
– calcul de Primitives,
– équations différentielles du premier ordre.

2.4 Enseignement durant le Post-Doc (70 HeqTD)

2.4.1 Tableau récapitulatif des enseignements à l'Université Catholique de Louvain

• **2005 -2006 Licence (UCL, Belgique)**

Intitulé de la composante de Module	Niveau	Cours (h)	TD (h)	TP (h)
Mathématiques	Lic 2		10	
Mathématiques	Lic 2		60	
Total :			70	

***Total* : 70 heures équivalents TD**

Tableau 2.3 : *Résumé des enseignements effectués en tant que Post-Doc*

2.4.2 Description détaillée des enseignements à l'Université Catholique de Louvain

• **Enseignements Licence deuxième année : Mathématiques**
Ce cours (10h) est destiné à des ingénieurs de seconde année, le but étant qu'ils maîtrisent le raisonnement combinatoire dans les sujets classiques de dénombrement, de discuter et résoudre des récurrences linéaires de forme simple, et de mettre en application les notions fondamentales de la théorie des graphes.
Ce cours de mathématiques discrètes est divisé en trois parties :

– dénombrement et récurrences : ensembles et sous-ensembles (nombres binomiaux) ; fonctions et partitions ; fonctions génératrices ; récurrences linéaires à coefficients constants.
– structures algébriques : monoïdes et groupes ; anneaux ; fonctions booléennes.
– graphes : notions de base ; connexité et arbres ; circuits eulériens et cycles hamiltoniens, couplages et recouvrements, chemins de coût minimum.

Un groupe de TD était à ma charge.

• **Enseignements Licence deuxième année : Mathématiques**
Ce cours (60h) vise à présenter les outils de base de l'algèbre et de l'analyse, dans l'optique de leur utilisation dans les sciences naturelles et d'une formation au raisonnement mathématique. Le cours comporte quatre parties :

– calcul différentiel et intégral (Nombres et fonctions, Fonctions d'une variable réelle, Nombres complexes, Intégration et primitivation),
– calcul matriciel, calcul vectoriel,
– fonctions de plusieurs variables réelles,
– introduction aux équations différentielles.

Je pilote deux groupes de TD.

2.5 Enseignement en tant que Maître de Conférences ($\simeq$ 240 HeqTD/an)

2.5.1 Tableau récapitulatif des enseignements à l'Université Claude Bernard Lyon 1

Formation		Cours	TD	TP	Eq. TD
Polytech'Lyon 3A/ ISTIL 1ère année	Automatique		**16**	12	28
	Mathématique mise à niveau	6	6		15
	Méthode Numérique de Base	16	16		40
Polytech'Lyon 4A/ ISTIL 2ème année	Automatique	15	30		52.5
	Plans d'expériences	10	25		40
	Encadrement de stage (5 mois)				
Polytech'Lyon 5A/ ISTIL 3ème année	Bond Graph	**15**	**15**		37.5
	Projet/ Travaux Pratiques			45	45
	PFE				
	Encadrement de stage (6 mois)				
M2R GSA	Tutoriales EEA	2			3
				Total :	$\simeq$240h

Tableau 2.4 : *Résumé des Enseignements effectués en tant que MCU*

En gras les cours qui n'existent plus depuis 2012 ou qui ne sont plus au programme dans la nouvelle mouture sont en gras, les cours en plus que j'ai pris en charge en 2012 vis-à-vis du nouveau programme sont en surlignés.

• **2006-2012** : Mon service enseignement est principalement dédié à l'antenne del'ISTIL située à Roanne, qui prend en charge la formation des ingénieurs de la filière Systèmes Industriels (SI) en deuxième et troisième année. Si le reste de ma charge d'enseignement se déroule à l'ISTIL Lyon, mes fonctions d'enseignante m'ont amené à prendre en charge des activités d'encadrement dans le **master de recherche Génie des Systèmes Automatisés**.

• **2012-2013** : Depuis janvier 2012, l'ISTIL est devenu Polytech'Lyon, et de nombreux changements ont eu lieu. La plus importante pour la filière SI est le déplacement de la 1ère année qui s'appelle maintenant 3A (l'école ayant une classe préparatoire intégrée) sur Roanne (anciennement la 1ère année se faisait sur Lyon, les 2ème et 3ème année sur Roanne ; depuis la rentrée 2012-2013 les 3A, 4A et 5A se font sur Roanne).

• **2013-2014** : *Délégation CNRS au GIPSA-lab*, Grenoble. J'ai effectué 89,6 HeqTD (97.6 heures réelles).
Dans ce cadre, j'ai été conviée en tant que professeur invité à donner un **cours de Master 2 recherche (M2R) en Automatique à l'INP de Grenoble.**

• **2014-2015** : Décharge d'un demi-service liée à mon congé maternité. Pour des raisons personnelles, liées à mon bébé, j'effectue mon 1/2 service sur Lyon.
J'ai donc repris certains cours que j'effectuais auparavant sur Lyon auprès des filières

Mécanique et nouvellement Informatique (Automatique continue, Méthodes Numériques de Base (MNB)). Je me suis également investie dans l'encadrement de projets pour l'année préparatoire universitaire (AUP) de l'UCBL.

2.5.2 Description détaillée des enseignements à l'Université Claude Bernard Lyon 1

• **Bond Graph** (37.5 HeqTD) en 3ème année SI, Roanne (5A Polytech) : ce cours porte sur le language bond graph basé sur les notions de puissance, et des variables d'efforts et de flux. Très utilisé dans l'industrie automobile, le Bond Graph prend tout son sens en système industriel. Je gère seule le cours et j'ai créé tous les supports du cours magistral que des TD, de la création des feuilles d'exercices portant sur les bonds graphs et l'utilisation du logiciel 20-sim, ainsi que de l'examen.

• **Cours d'Automatique continue** (28 HeqTD) pour les filières mécanique et systèmes industriels en première année d'ISTIL Lyon (3A Polytech), en collaboration avec B. Maschke, en charge de la partie cours magistral. Depuis 2011, j'ai repris seule ce cours sur Roanne, en assurant les cours magistraux, les travaux dirigés, les examens. J'ai à cette occasion adapté les supports pour la filière systèmes industriels. L'automatique continue est un premier socle pour l'introduction de l'automatique discrète en 4A, très utile pour l'aspect robotique et programmation automate.
NB : cette année 2014-2015, j'ai repris les cours initiés par mon collègue B. Maschke sur Lyon.

• **Cours d'Automatique discrète** (52.5 HeqTD) en 2nd année SI, ISTIL Roanne (4A Polytech'Lyon), je suis seule responsable tant pour la partie cours que TD. J'en assure toute l'intendance (support de cours, TD, examens, correction). Ce cours porte sur les systèmes discrets, l'échantillonnage, et l'élaboration de commande en temps discret, etc...

• **Cours de Méthodes Numériques de Bases** (40 HeqTD) en Tronc commun de la première année de l'EPU Lyon (2011-2012), en collaboration avec Naima Débit, responsable du module et en charge de la partie cours magistral. Depuis 2012, j'ai repris seule ce cours sur Roanne, en assurant les cours magistraux, les travaux dirigés, les examens. J'ai à cette occasion adapté les supports pour la filière systèmes industriels. Ce cours porte sur le passage du calcul analytique/formel au calcul numérique avec tous les problèmes et enjeux s'y rapportant : trouver les bonnes méthodes et les adapter aux types de problèmes traités.
NB : cette année 2014-2015, j'assure les travaux pratiques sur Lyon pour la filière Informatique, et Informatique par alternance.

• **Cours de Plans d'expérience** (40 HeqTD) en 2^{me} année SI, ISTIL Roanne (4A Polytech), je suis seule responsable tant pour la partie cours que TD, j'ai créé tous les supports nécessaires à cet enseignement. Ce cours porte sur la planification d'expériences pour l'amélioration tant qualité, quantitatif ou du coût d'un ou plusieurs produits.
NB : depuis 2012, j'ai laissé ce cours avec tous ses supports à un nouveau collègue PRAG attaché à Roanne.

• **Travaux pratiques** (45 HeqTD) sur platines de type industriel pour les étudiants de SI 3ème année (5A Polytech). J'ai mis en place une série de TP (Vision, API, IAPV, BUSCAN, GMAO, CAN, CNA, Robotique, ...) dans le but de permettre à nos étudiants de manipuler des outils auxquels ils devront faire face par la suite dans l'industrie.
Les TP Vision, Automate Programmable Industriel, BUSCAN ont été monté en collaboration avec un collègue PAST.
Les TP IAPV, CAN, CNA, GMAO, Robotique ont été montés sous ma seule responsabilité. Le TP dit robotique a pour but d'aider à la conception du robot dans le cadre du concours EUROBOT (concours international de robotique ouvert aux amateurs auxquels plusieurs promotions participent).

2.5.3 Cours effectués en tant que chercheur

• **Cours invités aux JD-JN MACS 2011** (3 HeqTD) avec Emmanuel Trélat, Yann Le Gorrec, Laurent Lefèvre, Valérie Dos Santos Martins.
Course on Control of Distributed Parameter Systems
C'est un cours qui a été réalisé dans le cadre des Journées Doctorales organisées par le Groupe de Recherche MACS.
L'approche proposée dans ce module est une approche dite "système" des EDP dans laquelle on considère des systèmes ouverts commandés par le biais de variables internes ou à la frontière. L'objectif est de fournir les bases des outils mathématiques et des techniques de modélisation permettant de traiter efficacement ce type de systèmes à paramètre. Les problèmes de modélisation, d'analyse d'existence de solution, de commandabilité, de stabilisation sont abordés.

• **Tutorial pour l'école doctorale EEA** (Electronique, Electrotechnique, Automatique) 2010-2011-2012 (3 HeqTD)
Commande des systèmes régis par des EDP dans les cas linéaire et non linéaire - Coordonnées et invariants de Riemann.
Mots-clés : systèmes à paramètres distribués, dimension infinie, commande, systèmes linéaires et non linéaires
Un survol des propriétés des invariants de Riemann sera présenté ainsi que leur utilisation pour construire une commande stable en dimension infinie. Dans un premier temps, ce problème sera abordée pour des systèmes linéaires, en utilisant deux approches ; l'approche Lyapunov et la structure d'invariance propre aux coordonnées Riemanniennes. L'extension de ce deuxième cas pour des systèmes non linéaires sera vue en deuxième partie.

• **Cours de M2R Master MiSCIT - Master in Systems, Control & IT Industrial Process Automation / école doctorale EEATS, INP de Grenoble** 2014 (15 HeqTD)
Hyperbolic systems, from the statement to the stability
Ce cours vise à présenter les différents types d'équations dites hyperboliques, les problèmes numériques associés, les propriétés mathématiques des solutions, et quelques applications récentes en contrôle (incluant des exemples en simulation et expérimention).

2.5.4 Réalisation de supports d'enseignement

• Auteur de différents supports de Cours, TD et TP :
Fascicule de Cours/TD de Mathématiques mise à niveau, 3A Polytech, 12 pages
Support de Cours/TD d'Automatique Discrète, 4A Polytech, 120 diapositives
Support de Cours/TD Plans d'expériences, 4A Polytech, 120 diapositives
Support de Cours/TD Bond Graph, 4A Polytech, 100 diapositives
Support, installation des platines des Travaux Pratiques

• Co-auteur de différents supports de Cours, TD et TP :
Support de Cours/TD de méthodes numériques de base, 3A Polytech, 100 pages
Support de Cours/TD d'automatique continue, 3A Polytech, 80 pages

Pour tous les cours cités précédemment, j'ai pris en charge la rédaction des examens, de leur surveillance et de la correction.

• Réalisation de supports de cours spécifiques :
Support de Cours pour le JN-JD MACS (3h de cours)
Support de Cours pour les tutoriaux de l'école dosctorale EEA, UCBL/INSA/ECL (2h de cours = 3 HeqTD)
Support de Cours/TD/évaluation pour le M2R Master MiSCIT -Master in Systems, Control & IT Industrial Process Automation/école doctorale EEATS, INP de Grenoble (15h de cours = 22.5 HeqTD)

2.5.5 Encadrement de 5 stages en entreprise

• **Mars-juillet 2007** :
- Huan ZOU (5A), IMAGINE (42000 Roanne).
Mission : création de 3 modèles d'amortisseur de suspension d'avion avec l'aide de Michel LEBRUN et d'un rapport de la démonstration pour analyser et expliquer les modèles (3ème modèle à Shanghai en Chine).

• **Mars-juillet 2010** :
- Damien ZAMARRENO (5A) : Révillon Chocolatier, filiale du groupe Soparind Bongrain.
Mission : maintenance améliorative, amélioration continue et travaux neufs pour résoudre les problèmes et déficits de certains postes de travail en assurant la conduite de projets d'investissements.

• **Mars-juillet 2011** :
- Amel RAHMOUN (5A) : entreprise Révillon Chocolatier, filiale du groupe Soparind Bongrain,
Mission : chargée de projets de maintenance améliorative, amélioration continue et travaux neufs.

• **Mars-juillet 2012** :
- Robin BROUILLARD (5A) : entreprise Révillon Chocolatier, filiale du groupe Soparind Bongrain
Mission : chargée de projets de maintenance améliorative, amélioration continue et tra-

vaux neufs.

• **Mars-juillet 2013** :
- Ricardo KIAKAYAMA (5A) : IRIDIUM, Vaulx en Velin
Mission : Gestion de projets d'automatismes.

2.5.6 Suivi de stages en entreprise

Suivi de 12 stages de 5$^{\text{ème}}$ année :

Année	Nom	Entreprise	Sujet
Mars-juillet 2007	Isidro XAVIER-MARTINS	JTEKT AUTOMOTIVE LYON (69540 Irigny)	Intégration de capteurs, mise en place sur bancs d'essais
Mars-juillet 2008	Fousseny KONE	ARCELOR MITTAL (57283 Maizieres les Metz)	Etude des profils de laminage
Mars-juillet 2009	Simon BONNARDEL Yann CADILLON	SNCF, unité voie OUEST LR (11100 Narbonne) SNCF, unité INFRA OUEST (73026 Chambery)	Elaboration d'un système optimisé de suivi des processus à risques majeurs Création d'installation Permanentes
Mars-juillet 2010	Michel BACHELET	Enterprise III, (LaBelle, Florida)	Study on the Biofuels land use implication
Mars-juillet 2011	Sonia ANICHE Laurent WATON	SOITEC Parc Technologiques des Fontaines Infra SNCF, Paris	Modéliser des étapes de production dans le MES par la gestion de leurs données techniques Elaborer des ratios d'utilisation des ressources engins et main d'oeuvre
Mars-juillet 2012	Camille BAUDIN Pierre GAYAT de WECKER Cédric BERNACHOT	SEMITAG, Grenoble LEGRAND France - Site de Châlus Néolution, Mions	Analyse critique de la chaine de traction des tramways TFS G4 Application de la démarche 5S en atelier moulage Simulation de flux ; du modèle à l'exploitation
Mars-juillet 2013	Johan ALLARDET Arthur José BEL	EKIUM, Lyon Cerebromix, Brésil	Automatisation et supervision d'une usine de traitement de déchets Gestion de projet des appareils à la pression appliquée dans les établissements industriels

Suivi de 5 stages de 4ème année :

Année	Nom	Entreprise	Sujet
Mars-juillet 2011	Sarra ALOUI	ALSTOM Transport Villeurbanne	Unification des systèmes driver AGATE
Mars-juillet 2012	Ricardo KIA-KAYAMA	Auto Châssis International (ACI) Villeurbanne	Standardisation des opérations de maintenance autonome
Mars-juillet 2013	Maxime BELIN	ATECAL, Vaulx en Velin	Assistance à l'intérieur de calculateur dans le cadre du projet RJH (Cadarache)
Septembre 2013 - janvier 2014	Grace MAKOUNDZI-WOLO	Eurocopter- Airbus Helicopters, Marignanne	Mise en place d'un print dashboard
	Rémy PAOLI	ARTELIA Eau & Environnement, Echirolles	Rédaction du Dossier de Consultation des Entreprises

2.6 Responsabilités collectives liées aux activités d'enseignement

Une école vit sur ses membres, c'est à dire ses enseignants, ses élèves et le personnel administratif. Si l'un de ses trois membres est absent, c'est toute la gestion de l'établissement qui périclite.
En dehors, des activités "normales" de tout enseignant, à savoir participer aux jurys de stages (stage ouvrier des 3A lorsqu'ils existaient, 4A et 5A), au recrutement (entretiens de l'école puis du réseau Polytech, concours GEIPI) que ce soit pour la "prépa intégrée" ou pour l'entrée en troisième année, je me suis investie dans les différentes instances de l'école (conseil de filière, conseil de la vie étudiante) et de l'université (conseil étudiant de la vie universitaire CEVU).

2.6.1 Responsable de modules - Polytech'Lyon

Depuis mon arrivée, j'ai été nommée reponsable des modules d'enseignements suivants :

– Responsable du module Méthode Numérique de Base en 3A,
– Automatique, Plan d'expérience en 4A,
– Modélisation par Bond Graph et TP Plateforme et logistique en 5A.

J'en assure le bon fonctionnement tant d'un point de vue logistique (gestion des emplois du temps, examens, retour des notes, intervenants...), que matériel (renouvellement de matériel, des licences..).

2.6.2 Membre du conseil de filière SIR - Polytech'Lyon

En 2011, des conseils de filières ont été mis en place afin d'officialiser un concept qui existait pour certaines filières.
Le but de ces conseils est de discuter de la filière, de son devenir, mais aussi de débattre en amont du jury final des notes obtenues par les étudiants en difficulté dans les Unités d'Enseignement (UE). En effet le mode de fonctionnement des jurys ayant changé dans

une optique pyramidale (chaque responsable d'UE renvoie ses notes avec une appréciation au conseil de filière qui fait des propositions au conseil de l'école qui statut).
Ce conseil élabore également les compte-rendus de filière pour la Comission des Titres de l'Ingénieur, CTI, afin d'obtenir sa certification.

2.6.3 Membre élu du CEVE - Polytech'Lyon

Le rôle du Conseil des études et de la vie étudiante (CEVE) est de discuter des orientations pédagogiques de l'école, statuer sur les réformes à entreprendre ou pas, de son évolution vers l'international et des prérogatives nécessaires à son accroissement et au bien-être des ses étudiants.
Il épaule le CoDir dans les décisions à prendre concernant les modalités de contrôle, la trame des cours dispensés, etc...
Appelée initialement commission pédagogique, puis CEVE, j'en suis membre depuis 2007.

2.6.4 Membre élu du GTVE, UCBL Lyon

Le groupe de travail de la vie étudiante (GTVE) a pour mission de promouvoir, d'étudier, de participer et de réaliser en partenariat avec les services compétents, les actions concourrants à une amélioration de la vie étudiante.
C'est un groupe de travail du CEVU (conseil des études et de la vie universitaire), dont je suis un membre élu depuis 2012, qui se réunit mensuellement pour discuter des actions relatives aux informations vis-à-vis des étudiants, des activités culturelles, des campagnes de sensibilisation et de prévention, de la gestion des fonds de solidarités des initiatives etudiantes (FSDIE).

CHAPITRE 3

Synthèse des Activités de Recherche

"*Chercher n'est pas une chose et trouver une autre, mais le gain de la recherche, c'est la recherche même.*"

de Saint Grégoire de Nysse, théologien (331-394), Extrait de Homélies sur l'Ecclésiaste .

Sommaire

Ce chapitre présente dans un premier temps les différentes actions de recherches menées autour des thèmes de la stabilité des systèmes à paramètres distribués décrits par des équations aux dérivées partielles (EDP) hyperboliques.
Je développe ensuite le travail d'encadrement de jeunes chercheurs, les responsabilités collectives d'encadrement et donne la liste complète des contributions scientifiques.

3.1 Résumé des activités de Recheche

Mon activité de recherche se décline en trois thèmes :
- stabilité par inégalités linéaires d'opérateurs.
- stabilité et contrôle de procédés à frontière variable,
- systèmes hamiltoniens à port : invariants, lien des structures Hamiltoniennes et Riemanniennes...

3.1.1 Stabilité par inégalités linéaires d'opérateurs

Ce projet prend place au sein du LAGEP dans le domaine de l'automatique et plus précisément sur le développement des techniques de commande robuste dédiées aux systèmes de dimension infinie. Le challenge consiste à contrôler la ou les variable(s) d'un système à paramètres distribués décrits par des EDP hyperboliques, sur la toute la plage admissible en garantissant d'un point de vue théorique la stabilité de l'ensemble. L'approche est illustrée sur la commande de canaux d'irrigation.
L'objectif est la synthèse de lois de commandes robustes pour des systèmes de dimension infinie en étudiant plus particulièrement les techniques connues de représentations Multi-Modèles en dimension finie. Il s'agit donc d'étudier la transposition non-immédiate des résultats théoriques de haut niveau sur l'étude de la stabilité et de la commande des systèmes en dimension finie représentés par des Multi-Modèles vers les systèmes de dimension infinie, toujours en utilisant la représentation Multi-Modèles. Les Multi-Modèles sont en effet particulièrement bien adaptés aux systèmes de grande dimension tels les canaux d'irrigation qui considèrent plusieurs points de fonctionnement.
Ce travail a réellement débuté par le master de ***Mamadou Diagne*** en 2008-2009 en collaboration avec **Mickael Rodrigues** (MCU université Lyon 1, LAGEP), et qui a permis de définir la problématique, les verrous techniques/scientifiques et d'initier les premiers résultats que nous avons repris et améliorés [CIFA-2010].
Cette première approche s'appuie sur l'étude d'une commande multi-modèles de type intégral sur une structure de commande frontière par modèle interne que nous avons élargie à une commande proportionnelle intégrale (PI) dans des cas spécifiques qui nous permettent de considérer des outils particuliers de la dimension finie en dimension infinie [AMCS-2012] [18IFAC-2011].
Le travail de Master de ***Yongxin Wu*** en 2012, basé sur une généralisation des techniques précédemment utilisées par Mickael Rodrigues en dimension finie [ECC-2013], nous a permis d'aborder une nouvelle approche et de définir des outils dédiés à la dimension infinie. Le concept de LOI pour **Linear Operators Inequality**, le pendant des LMI (Linear Matrix Inequality), a pu être défini permettant ainsi de valider d'un point de vue théorique l'approche effectuée cette fois en dimension infinie sans contraintes particulières [TCST-2013].
L'ensemble de ces travaux a été validé en simulations, démontrant l'amélioration concrète

des performances des commandes implémentées sur différents canaux (canal expérimental, canal réel). J'ai également validé cette approche expérimentalement. Recemment nous avons également développé une commande multi-modèles dont les poids sont des fonctions lisses et non plus binaires, i.e. 0 ou 1 [en soumission].

3.1.2 Stabilité et contrôle de procédés à frontière variable

Suite à son master avec moi, nous avons avec **Mamadou Diagne** proposé un sujet de thèse en commun avec la thématique de l'équipe Systèmes Dynamiques dont le sujet est "Commande d'un procédé à frontières variables de dimension infinie, non linéaire : application aux procédés d'extrusion". L'équipe de recherche "Dynamique des procédés" intervient à ce titre dans des projets du pôle de compétitivité "Chimie et Environnement", et participe en particulier au "working package" "intensification des Procédés" sur un projet de recherche concernant la modélisation de procédés d'extrusion réactive, en collaboration avec la société Rhodia. Ce projet fait suite à une première étude menée dans le cadre d'un Contrat Programme de Recherche (CPR) Matériaux en collaboration avec le Laboratoire des Matériaux Plastiques et Biomatériaux (UMR CNRS 5627). L'intensification des procédés est un sujet crucial pour l'industrie chimique dans le cadre du développement durable.
Ces procédés sont caractérisés par une géométrie très complexe où se déroulent des réactions diverses suivant les différentes zones du réacteur, et par des objectifs de qualité de produits difficilement mesurables. Ces zones sont mouvantes suivant le flux de matières et mènent à un modèle dynamique constitué de systèmes à paramètres distribués à frontière mobile. La commande de tels systèmes à frontière mobile est peu traitée dans la communauté des automaticiens actuellement et nécessite le développement de méthodes nouvelles adaptées aux systèmes à paramètres distribués à frontière mobile.
Le sujet de thèse proposé traite des systèmes à paramètres distribués dont le domaine spatial est divisé par une interface mobile et est motivé par le problème de la commande des procédés d'extrusion. Les propriétés dynamiques de divers modèles ont été analysées [CDC-2011] [JESA-2011] [CIFA-2012], puis la commande d'un modèle bizone a été développée en utilisant un modèle de système à retard dépendant de la variable de commande. En parallèle, une commande PI implémentée dans une structure de commande prédictive (IMBC) a été développée permettant de contrôler la température en des points choisis du procédé [MTNS-2014] avec un modèle prenant en compte la viscosité de la matière. Des systèmes de deux lois de conservation hamiltoniens à ports avec interface mobile ont également été traités en étendant leur formulation hamiltonienne aux fonctions caractéristiques des domaines spatiaux séparés par l'interface.

3.1.3 Les systèmes hamiltoniens à port : invariants, lien des structures Hamiltoniennes et Riemanniennes

Cette thématique phare de l'équipe DYCOP, est au coeur de presque tous les sujets traités d'un point de vue de l'automatique par mes collègues de l'équipe.
Mes premiers pas sur les Hamiltoniens se sont faits dans le cadre de l'ANR PARADE et de la thèse de Redha Moulla et en parallèle via le sujet de master de **Yves Hagedorn**, dont le stage portait sur l'"Etude de stabilité non linéaire d'un écoulement unidirectionnel par fonction de Lyapunov : une approche hamiltonienne" effectué sous ma direction. Les travaux entamés se sont développés en collaboration avec Bernhard Maschke et Yann Le Gorrec, puis seule. Ils traitent des systèmes Hamiltoniens à port de

dimension infinie (structure de Dirac) et des propriétés d'invariance de certains jeux de variables. Initialement, nous sommes partis sur les invariants de Casimir, puis très rapidement nous avons fait le lien avec les invariants de Riemann. Nous avons démontré que les différentes structures pouvaient être liées par des transformations de type Cayley [NOLCOS-2010] [NHM-2008] [Irr-Wkshp-2008]. J'ai par la suite étendu ces travaux et démontré que l'on pouvait également faire le lien avec le théorème du Petit Gain [MTNS-2014] [IEEE-TAC en soumission].
Parallèlement, la thématique de l'ANR PARADE concerne la semi-discrétisation et la réduction préservant la structure hamiltonienne à ports, des systèmes de lois de conservation hamiltoniens dissipatifs et a donné lieu aux travaux de thèse de A. BAIIU, de H. PENG et de R. MOULLA.

Je me suis également investie dans l'approche géométrique, prolongation des travaux sur une classe de systèmes non linéaires adaptés à la Thermodynamique Irréversible dans le sens où ils sont définis sur un espace d'état muni d'une forme de contact associée à l'équation de Gibbs : les systèmes de contact entrée-sortie. Les travaux de thèse d'Audrey FAVACHE ont porté sur une formulation alternative de modèles de procédés où la fonction génératrice de la dynamique, la fonction hamiltonienne, a la dimension d'une entropie par unité de temps et représente un flux d'entropie virtuel, et sur leur propriétés dynamiques [IEEE-TAC-2009] [IEEE-TAC-2009].
Un pan de ses travaux sur les approches géométriques a porté sur les graphes de liaison. Nous avons étudié l'application du formalisme des graphes de liaison au génie des procédés. Il s'agit d'une extension du formalisme de type réseau électrique qui permet de représenter des systèmes multi-physiques de manière unifiée, indépendamment du domaine physique ou bien de la nature localisée ou distribuée du modèle. Ils se distinguent des modèles usuels par l'expression systématique du bilan d'entropie et l'écriture de modèles algébro-différentiels implicites basés sur l'utilisation de paires de variables conjuguées permettant de traiter aisément le cas des systèmes ouverts et de leur interconnexion. L'usage de ce langage graphique et des logiciels existants permet de garantir une meilleure réutilisabilité des sous-modèles dans un contexte donné et donc une meilleure gestion des travaux de modélisation. Nous avons montré comment utiliser ce formalisme en Génie des Procédés [CIFA-2008-1] [CIFA-2008-2].

Dans le cadre de l'ANR HAMECMOPSYS, nous travaillons en collaboration avec Thomas Helie, CR1 IRCAM, sur des problèmes d'acoustique et de découplement des ondes dans une section non droite, conique, ou courbe d'un tube acoustique avec écoulement. Le problème vient du fait que le tube est à section variable donc qu'il y a couplage des ondes. Nous essayons de découpler via les structures Hamiltonniennes et/ou d'invariants. Ce sujet est en cours et n'a pas pour le moment abouti.

3.2 Encadrement de Jeunes Chercheurs

3.2.1 Chiffres

• 2 thèses soutenues et 1 participation à une thèse.
• 5 masters M2R dont 2 ont été pris en thèse au sein de l'équipe DYCOP, 1 est parti aux USA en poursuite d'études, 1 est en cours.

3.2.2 Thèses soutenues

• **Mamadou Diagne**, thèse soutenue le 26 juin 2013 devant l'Université Claude Bernard Lyon1. Actuellement en post-doctorat au Département de Mécanique and Ingénierie Aérospatiale de l'université de Californie, San Diego, USA avec Miroslav Krstic sur la commande d'imprimante 3D.
Titre : "Etude des systèmes de lois de conservation à interface mobiles : application à un procédé d'extrusion".
Jury :
M. Jean-Michel CORON, Professeur des Universités Université Pierre et Marie Curie, Paris
Mme Françoise COUENNE, Chargée de Recherche LAGEP Villeurbanne
M. Michael DI LORETO, Maître de Conférences INSA de Lyon
M. Didier GEORGES, Professeur des Universités Institut Polytechnique de Grenoble
M. Yann LE GORREC, Professeur des Universités École Nationale Supérieure de Mécanique et des Microtechniques de Besançon
M. Bernhard MASCHKE, Professeur des Universités Université Claude Bernard Lyon 1
M. Nicolas PETIT, Professeur Mines ParisTech
M. Hans ZWART, Professeur Université de Twente - Pays Bas
Taux d'encadrement 30% avec B. Maschke, Prof. Université Lyon 1 et F. Couenne, CR CNRS.
2 article de journal dont 1 commun [JESA-2011]
4 articles de congrès dont 2 communs [CIFA-2012] [CDC-2011]

• **Audrey Favache,** soutenue le 03 septembre 2009 devant l'université Catholique de Louvain, employée en tant que chercheuse contractuelle FNRS - Université Catholique de Louvain.
Titre : "Thermodynamique et commande des procédés : l'entropie comme outil de synthèse et d'analyse des systèmes de commande".
Jury :
M. Bernhard MASCHKE, Professeur des Universités, Université Claude Bernard Lyon 1
M. Christian JALLUT, Professeur des Universités, Université Claude Bernard Lyon 1
M. Denis DOCHAIN, Professeur, Université Catholique de Louvain
M. Joseph WINKIN, Professeur, Université de Namur (Belgique)
M. Christian BAILLY, Professeur, Université Catholique de Louvain
M. Georges BASTIN, Professeur, Université Catholique de Louvain
M. Antonio ALONSO, Professeur, Instituto de Investigacioes Marinas (CISC)
M. B. Erik YDSTIE, Professeur, Carnegie Mellon University
Thèse en collaboration. Encadrants : B. Maschke, Prof. Université Lyon 1 et D. Dochain, Prof. Université Catholique de Louvain (Belgique).
1 article de journal en commun [IEEE-TAC-2009]
1 article de congrès en commun [ECC-2009]

3.2.3 Master 2 de Recherche

Les étudiants du master GSA sont peu nombreux et en général la plupart suivent un double cursus INSA/ ECL et Master2. Pour les étudiants que j'ai encadrés, le travail a commencé en amont par une étude bibliographique (UE obligatoire du Master) de novembre à février.

• Mars-juillet 2007 : Encadrement de **Yves Hagedorn**, à 100%
Sujet : " Etude de stabilité non linéaire d'un écoulement uni-directionnel par fonction de Lyapunov : une approche Hamiltonienne".
Poursuite d'études aux US, puis reprise de l'entreprise familliale.

• Mars-juillet 2008 : Encadrement de l'étudiant **Samir Chabou**, co-encadré avec Mickael Rodrigues à 50%, maitre de conférences (LAGEP).
Sujet : " Commande et l'analyse des systèmes hydrauliques par une approche multi-modèles en dimension finie et/ou infinie ".

• Mars-juillet 2009 : Encadrement de l'étudiant **Mamadou Diagne**, co-encadré avec Mickael Rodrigues à 50%, maitre de conférences (LAGEP).
Sujet : " Commande et l'analyse des systèmes hydrauliques par une approche multi-modèles en dimension finie et/ou infinie " .
Poursuite en thèse sous la direction du professeur Bernhard Maschke, et le co-encadrement de Françoise Couenne et de moi-même.
2 publications liées au stage : **1 congrès** [CIFA 2010], **1 journal** [AMCS 2012]

• Mars-juillet 2013 : Encadrement de l'étudiant **Yongxin WU**, co-encadré avec Mickael Rodrigues à 50%, maitre de conférences (LAGEP).
Sujet : "Commande et analyse des systèmes hydrauliques par une approche Multi-Modèles en dimension infinie" .
Poursuite en thèse, sous la direction seule du professeur Bernhard Maschke sur les systèmes Hamiltoniens, seul sujet prioritaire de l'équipe.
3 publications liées au stage : **1 congrès** [ECC 2013], **1 journal** [IEEE TCST 2013] **1 séminaire invité** [SAR-2012]

• Mars-juillet 2015 : Encadrement de l'étudiant **Alexandre de TERRAND-JEANNE**, à 100%.
Sujet : "Control methodology for large scale electro-mechanical systems" .
En cours...

3.3 Responsabilités collectives liées aux activités de recherche

3.3.1 Membre d'un comité technique de L'IFAC

En 2011, renaissait le comité technique 2.6 sur les "Systèmes à Paramètres Distribués" après quelques années de pause. Pour cela les membres des différents groupes de travail associés se sont réunis, et nous avons relancé le T.C. 2.6. Le rôle de ce dernier est de faire vivre ce groupe via des animations, comme la création de congrès spécifique (IFAC CPDE), de sessions invitées (MTNS, World IFAC,...) et d'en assurer l'intendance (pour ma part je fais partie de l'IPC, je suis éditeur technique associée, présidente de session dans différentes conférences...).

3.3.2 Membre élu au comité consultatif de la 61ème section de CNU

Depuis novembre 2008, je suis membre élue au comité consultatif de la 61ème section du CNU à l'Université Lyon 1. Ce comité s'est vu confier le rôle d'organiser le recrutement de plusieurs professeurs et maîtres de conférences. Nous organisons chaque année l'expertise et le classement des demandes d'ATER en 61ème section à l'Université Lyon 1. J'ai participé également à plusieurs recrutements de MCU tant dans la section CNU 61 que dans des sections connexes.

3.4 Projets

3.4.1 Projets déposés

Le dépôt de projet fait partie intégrante du travail de l'enseignant chercheur et je participe au dépôt d'un projet en moyenne par an.

En collaboration avec Mickaël Rodrigues, nous avons déposé à plusieurs reprises des ANR sur notre sujet commun (ANR blanc, JCJC), des PEPS, et un projet RPDOC. Nos démarches n'ont pas abouti.
• **2014** ANR CLEAN WATER, classé $13^{ime}/160$, seuls les 6 premiers ont été pris.
• **2012** ANR ARTHEMIS, ANR JCJC SIMI3 - Matériels et logiciels pour les systèmes de communications, pas de classement connu (7 projets financés sur 44 déposés).
• **2012** Projet Chante Philomèle, PEPS INS2I, pas de classement connu.
• **2011** Projet SYNPHOMINE, PEPS SIAR, pas de classement connu, 87 dossiers déposés.
• **2011** Projet SYNPHOMINES, PEPS INSIS, pas de classement connu.
• **2010** Projet E=$(MC)^2$, ANR JCJC, classé A mais pas de financements suffisants.
• **2010** Projet E : $(MC)^2$, ANR RPDOC, pas de classement connu.
• **2009** Projet SIROCO, ANR JCJC, pas de classement connu.

Je participe ou ai participé à deux projets ANR au sein de l'équipe DYCOP, pour lesquels ma contribution est scientifique et j'ai fait partie du comité de recrutement des étudiants en thèse pour le LAGEP.

3.4.2 ANR Technologie Logicielle "Numérique" PARADE

• **Date** : 2006 - 2010
• **Titre** : Algorithmes Numériques/Symboliques Parallèles de Résolution de Systèmes d'Equations Algébro-Differentielles (Parallel numerical Algorithms for Real time simulation of Algebraic Differential Equations systems)
• **Partenaires** : "CDCSP" (Centre pour le développement du Calcul Scientifique Parallèle) de l'Institut Camille Jordan (ICJ) - UMR5208 Un. Lyon 1-ECL-INSA-CNRS, "Dynamique des procédés : modèles, structures de modèles et commande" du Laboratoire d'Automatique et de Génie des Procédés - UMR 5007 Univ. Lyon 1-CNRS, le LIMOS (Laboratoire d'Informatique, de Modélisation et d'Optimisation des Systèmes) laboratoire commun de l'Université d'Auvergne Clermont I et de l'Université Blaise Pascal (UBP) Clermont II (UMR 6158, CNRS), l'équipe de recherche METALAU (Méthodes, algorithmes et logiciels pour l'automatique) de l'INRIA, institut national de recherche en informatique et

en automatique, IMAGINE Société de Services en Ingéniérie Systèmes (Roanne) et Siemens VDO Automotive SAS.
• **Thèmes** : diminution du temps de calcul par la parallélisation de systèmes d'EDOs ou d'EADs issus de la modélisation de systèmes physiques complexes par composantes fonctionnelles, réduction du temps de calcul, distribution des calculs sur plusieurs processeurs.

• Lors de mon arrivée en 2006, on m'a proposé d'intégrer ce projet initié en amont. Sur certains points le sujet me semblait intéressant, mais très loin de mon activité de recherche. Après avoir participé au projet et au recrutement du doctorant qui travaillerait dessus au LAGEP, je me suis retirée au bout d'un an du programme.

3.4.3 ANR Blanc HAMECMOPSYS

• **Date** : 2011-2015
• **Titre** : Approche Hamiltonienne pour l'analyse et la commande des systèmes multiphysiques à paramètres distribués)
• **Partenaires** : FEMTO-ST/AS2M, UMR CNRS, Besançon (Yann Le Gorrec), LAGEP, UMR CNRS 5007 (B. Maschke), Institut Supérieur de l'Aéronautique et de l'Espace (ISAE), Toulouse (D. Matignon), Université de Nancy, Institut Elie Cartan (M. Tucsnak)
• **Thèmes** : **systèmes hamiltoniens** à port pour l'interaction fluide-structure (implants cochlée, nano-manipulateur dans fluide, solides déformables dans fluide, extrudeuse), **la dissipation** (acoustique, aliage à mémoire de forme (hystérésis), système à frontière mobile, commande IDA-PBC et optimale).

• Mon implication dans cette ANR s'est faite en deux étapes. Il y a eu tout d'abord la thèse de Mamadou DIAGNE qui souhaitait poursuivre notre collaboration après son stage de master avec moi. Cela correspond au thème **systèmes hamiltoniens** à port pour l'interaction fluide-structure, et le procédé d'extrusion.

Le deuxième thème porte les problèmes d'ondes en acoustique, thème sur lequel je travaille en collaboration avec Thomas Helie, CR1 à l'IRCAM à Paris.

3.4.4 PAI Projets du programme de coopération franco-belge "Tournesol"

• **Date** : 2005 - 2006 et 2008 - 2009
• **Titres** : "Méthodes de discrétisation géométriques pour l'automatique des systèmes dynamiques ouverts en génie des procédés", et pour le second "Thermodynamique et commande des procédés : l'entropie comme outil de synthèse et d'analyse des systèmes de commande".
• **Partenaires** : LAGEP UMR CNRS 5007, CESAME/INMA UCL.
• Ces Projets d'Action Intégrée franco-belge sont réalisés en collaboration avec le Professeur Denis Dochain de l'Université Catholique de Louvain (Louvain-la-Neuve).
Le premier projet a permis de poursuivre la collaboration avec la professeur Georges Bastin. Le second projet a été le support pour les échanges d'enseignants-chercheurs autour de la thèse en co-tutelle de Audrey Favache.

3.5 Liste complète des contributions scientifiques

Les étudiants encadrés sont en italique.

3.5.1 Chiffres

Au 01 janvier 2015, sont actés **41 références** bibliographique et **22 séminaires** :
- 11 articles de journal
- 22 congrès Internationaux avec comités de lecture et actes publiés
- 3 congrès Internationaux sans comités de lecture
- 5 communications nationales
- 22 séminaires

3.5.2 Publications Internationales avec comité de lecture

- **11 Articles de journal :**

[TCST-2013] **DOS SANTOS MARTINS V.**, RODRIGUES M., *WU Y.*, "Design of a PI Control using Operator Theory for Infinite Dimensional Hyperbolic Systems", IEEE Transactions Control Systems Technology (**IEEE TCST**, IF :2.512), TCST-2013-0381, Vol 22, Issue 5, pp. 2024 - 2030, DOI 10.1109/TCST.2014.2299407.

[AMCS-2012] **DOS SANTOS MARTINS V**., RODRIGUES M., *DIAGNE M.*, "A Multi-Models approach of Saint-Venant's equations : a stability study by LMI", International Journal of Applied Mathematics and Computer Science (**AMCS**, IF : 1.390),Volume 22, Numéro 3, pp 539-550.

[JESA-2011] *DIAGNE M.*, **DOS SANTOS MARTINS V.**, COUENNE F., MASCHKE B., JALLUT C., "Méthodes numériques et applications des systèmes à paramètres repartis", Journal Européen des Systèmes Automatisés (**JESA**), VOL 45/7-10 - 2011, pp.665-691 (doi :10.3166/jesa.45.665-691)

[IEEE-TAC-2009] *FAVACHE A.*, **DOS SANTOS MARTINS V.**, DOCHAIN D., MASCHKE B., "Some Properties of Contact Structure Dynamical Systems", IEEE Transactions on Automatic Control (**IEEE TAC**, IF : 3.167) , Volume : 54 Issue :10, pp : 2341 - 2351, 2009

[NHM-2008] **DOS SANTOS V.**, LEGORREC Y., MASCHKE B., " A potential link between passivity condition and stability : application on shallow water equations", AIMS Networks and Heterogeneous Media (**NHM**, IF : 0.952), 4(2), 249-266, 2009

[IEEE-TCST-2007] **DOS SANTOS V.**, PRIEUR C., "Boundary control of open channels with numerical and experimental validations", IEEE Transactions Control Systems Technology (**IEEE TCST**, IF :2.512), Volume 16, Issue 6, Nov. 2008 Page(s) :1252 - 1264.

[Auto-2007] **DOS SANTOS V.**, BASTIN G., CORON J.-M., d'ANDREA-NOVEL B., "Boundary control with integral action for hyperbolic systems of conservation laws : Lyapunov stability analysis and experimental validation", ELSEVIER **Automatica** (IF :

3.132), Vol. 44(5), 2008, pp. 1310 - 1318, PII : S0005-1098(07)00447-5.

[CAO-2007] **DOS SANTOS V.**, BASTIN G., TOURÉ Y., "Regulation in Multireach Open Channels by Internal Model Boundary Control", Special Issue on : Control Applications of Optimisation - control and aeronautics, optimal control, control of partial differential equations, International Journal of Tomography & Statistics (**IJTS** IF : 0.23), pp 91-96, Vol 5, No. W07, winter 2007.

[e-STA-2004] **DOS SANTOS V.**, TOURÉ Y., CISLO N., "Régulation de Canaux d'irrigation : Approche par Contrôle Frontière Multivariable, et Modèle Interne d'EDP", revue **e-STA**, Sciences et Technologies de l'Automatique, Volume 1, N^{o}4 Quatrième trimestre 2004.

[SAMS-2003] **DOS SANTOS V.**, TOURÉ Y., "On the regulation of irrigation canals : multivariable boundary control approach by internal model", Taylor and Francis, journal Systems, Analysis, Modeling Simulations (**SAMS**).

[HYKE-2003] CORDIER S., BUET C., **DOS SANTOS V.**, "A Conservative and Entropy Scheme for a Simplified Model of Granular Media", Taylor and Francis, Transport Theory and Statistical Physics (**TTSP**, IF : 0.417), Volume 33, Issue 2, Avril 2004. HYKE 2003-021

• **22 Congrès Internationaux avec comités de lecture et actes publiés**

[MTNS-2014] **DOS SANTOS MARTINS V.**, "Link between Dissipativity Expressed in Riemann Coordinates and the Small Gain Theorem, Using the Hamiltonian Formulation", The 21st International Symposium on Mathematical Theory of Networks and Systems, **MTNS 2014**, invited session organized by Birgit Jacob, Kirsten Morris and Michael Demetriou, July 7-11, 2014, Groningen, The Netherlands (session invitée)

[ECC-2013] **DOS SANTOS MARTINS V.**, *WU Y.*, ABERKANE S., RODRIGUES M., "LMI & BMI Technics for the Design of a PI Control for Irrigation Channels", European Control Conference, **IEEE ECC** 2013, juillet, Zurich, Suisse

[CPDE-2013] **DOS SANTOS MARTINS V.**, "Control of a system a coupled PDEs with ODE in infinite dimension ; Application to an extrusion process", **IFAC CPDE 2013**, invited session "PDE Applications" organized by V. Dos Santos Martins, 1st IFAC Workshop on Control of Systems Governed by Partial Differential Equations, September 25-27, 2013, The Institut Henri Poincaré in Paris, France (organisatrice de session invitée)

[CIFA-2012] **DOS SANTOS MARTINS V.**, *DIAGNE M.*, COUENNE F., MASCHKE B., "Stabilité d'un procédé d'extrusion par deux systèmes d'équations d'évolution couplés par une interface mobile", Conférence Internationale Francophone d'Automatique (**CIFA 2012**), Conférence **IEEE**, Grenoble, (n^o 174) 2012

[18IFAC-2011] **DOS SANTOS MARTINS V.**, RODRIGUES M., "A Proportional Integral Feedback for Open Channels Control through LMI Design", 18th **IFAC World Congress**, Milan,Italy, September, 2011

[CDC-2011] *DIAGNE M.*, **DOS SANTOS MARTINS V.**, COUENNE F., MASCHKE B., "Well posedness of the model of an extruder in infinite dimension", 50th **IEEE Conference on Decision and Control** and European, Control Conference,Orlondo, FL, USA on December 12-15,(n^o 1926) 2011

[CIFA-2010] *DIAGNE M.*, **DOS SANTOS V.**, RODRIGUES M., "Une approche Multi-modèles des équations de Saint- Venant : une analyse de la stabilité par techniques LMI", Conférence Internationale Francophone d'Automatique (**CIFA 2010**), Conférence **IEEE**, Nancy, juin2010

[NOLCOS-2010] **DOS SANTOS V.**, LEGORREC Y., MASCHKE B., "Hamiltonian approach to the stabilization of systems of two conservation laws" (Paper ID : 95, Invited Paper), 8th **IFAC Symposium** on Nonlinear Control Systems, (NOLCOS 2010), Bologna, Italy, septembre 2010

[ECC-2009] *FAVACHE A.*, **DOS SANTOS MARTINS V.**, DOCHAIN D., MASCHKE B., "Stability Properties of Contact Structure Dynamical Systems", **IEEE Control Systems** Society Conference, ECC2009, European Control Conference, paper n^{o}423

[CIFA-2008-1] COUENNE F., **DOS SANTOS V.**, LEGORREC Y., MASCHKE B. "Structure de jonction pour les systèmes à paramètres distribués : cas des systèmes réversibles et irréversibles", **CIFA 2008**, n^{o}T18-I-07-0149, Bucarest, septembre 2008

[CIFA-2008-2] COUENNE F., **DOS SANTOS V.**, LEGORREC Y., MASCHKE B. "Approche géométrique pour l'analyse et la commande des systèmes de dimension infinie", **CIFA 2008**, n^{o}T10-I-41-0239, Bucarest, septembre 2008

[Irr-Wkshp-2008] **DOS SANTOS V.**, LEGORREC Y., MASCHKE B. "Passivity condition for stability : application on shallow water equations", Workshop on "Irrigation Channels and Related Problems", Maiori, Salerno, Italy, October 2008, (invited speaker)

[Nolcos-2007] **DOS SANTOS V.**, BASTIN G., CORON J.-M., d'ANDREA-NOVEL B., " Boundary control of Systems of Conservation Laws : Lyapunov Stability with Integral Actions ", 7th **IFAC Symposium** on Nonlinear Control Systems, Nolcos 2007, n^{o}50, Prétoria, South Africa, aout 2007

[SSSC-2007] **DOS SANTOS V.**, PRIEUR C., "Boundary Control of a channel : practical and numerical studies ", 3rd **IFAC Symposium** on System, Structure and Control, SSSC 2007, n^{o}145, Foz de Iguaçu, Brasil, Novembre 2007 (chairman de session)

[ICINCO-2007] **DOS SANTOS V.**, PRIEUR C., "Boundary Control of a channel :last improvements " ICINCO 2007 649, co-sponsored by **IFAC**, Angers, 9-12 mai 2007

[CAO-2006] **DOS SANTOS V.**, BASTIN G., TOURÉ Y., "Regulation in Multireach Open Channels by Internal Model Boundary Control", 13th **IFAC Workshop** on Control Applications of Optimisation, CAO'06, Cachan, Avril 2006

[Controlo-2006] **DOS SANTOS V.**, TOURÉ Y., "Internal Model Boundary Control of Hyperbolic system : Application to the Regulation of Channels", 7thPortuguese Conference on Automatic Control - **CONTROLO'2006**, Lisbon, Septembre 2006

[16IFAC-2005] **DOS SANTOS V.**, TOURÉ Y., MENDES E., COURTIAL E., "Multivariable Boundary Control Approach by Internal Model, applied to Irrigation Canals Regulation", 16th **IFAC World Congress**, Prague, Czech Republic, from July 4 to July 8, 2005

[CDC-ECC05] **DOS SANTOS V.**, TOURÉ Y., "Irrigation Multireaches Regulation problem by Internal Model Boundary Control", 44ème CDC-ECC'05, **IEEE Control Systems** Society Conference, Espagne, décembre 2005

[CIFA-2004] **DOS SANTOS V.**, TOURÉ Y., CISLO N., "Régulation de Canaux d'irrigation : Approche par Contrôle Frontière Multivariable, et Modèle Interne d'EDP", Conférence Internationale Francophone d'Automatique (**CIFA 2004**), Conférence **IEEE**, Tunisie novembre 2004 (3ème prix des doctorants).

[CDS-2003] **DOS SANTOS V.**, TOURÉ Y., "Regulation of Irrigation Canals : Multivariable Boundary Control Approach by Internal Model", Second **IFAC Conference** on Control Systems Design (CSD'03), International Federation of Automatic Control, Pologne 7-10 Septembre 2003, n^o abs-056-35-35.

[SSD-2003] **DOS SANTOS V.**, TOURÉ Y., "On the Regulation of Irrigation Canals : Internal Model and Boundary Control Approach", Second International Conference on Signals, Decision and Information Technology (**SSD'03**), **IEEE**, Tunisie 26-28 Mars 2003, n^o SSD-03-A-MD-50.

3.5.3 Publications et Communications sans comité de lecture

• **3 publications internationales**

[CDPS-2009] **DOS SANTOS V.**, LEGORREC Y., MASCHKE B., "A Hamiltonian approach to the stabilization of systems of two conservation laws", **IFAC Workshop** - CDPS 2009, Control of Distributed Parameter Systems, Toulouse, 20-24 juillet 2009

[CDPS-2007-1] **DOS SANTOS V.**, TOURE Y., SAU J. "Boundary control of a channel : Internal Model Boundary Control", **IFAC Workshop** - CDPS 2007, Control of Distributed Parameter Systems, Namur, Belgique, juillet 2007

[CDPS-2007-2] **DOS SANTOS V.**, PRIEUR C., SAU J. "Boundary control of a channel in presence of small perturbations : a Riemann approach", **IFAC Workshop** - CDPS 2007, Control of Distributed Parameter Systems, Namur, Belgique, juillet 2007

• **5 communications nationales dont une invitée :**

DOS SANTOS V., "Régulation de canaux d'irrigation : contrôle en temps réel d'Equations aux Dérivées Partielles hyperboliques", CEA-CAMNI, La mécanique des Fluides

Numérique, Institut Henri Poincaré, Paris, janvier 2006.

• **22 séminaires dont 5 invités :**

[SAR-2012] **DOS SANTOS V.**, RODRIGUES M., *WU Y.*, "Design of a PI Control using Operator Theory for Infinite Dimensional Systems : Application to Hyperbolic PDE", Groupe de travail Systèmes à Retards, Réunion commune EDP-SAR du 18-19 Septembre 2012.

[EDP-2012] **DOS SANTOS V.**, RODRIGUES M., "Une approche Multi-Modèles sur un système hyperbolique en dimension infinie", Groupes de Travail EDP, Valence, 6&7 février 2012.

[INSA-2009] **DOS SANTOS V.**, "Formulation Hamiltonienne pour la stabilisation de systèmes de lois de conservation non linéaires de dimension infinie : généralisation", séminaire Easy-Commande, INSA, Lyon 1, 28 mai 2009.

[IHP-2006] **DOS SANTOS V.**, "Régulation de canaux d'irrigation : contrôle en temps réel d'Equations aux Dérivées Partielles hyperboliques", CEA-CAMNI, La mécanique des Fluides Numérique, Institut Henri Poincaré, Paris, janvier 2006.

[Orsay-2005] **DOS SANTOS V.**, TOURÉ Y., "Lieu d'Evans : Outils de Synthèse de Contrôle en Dimension Infinie", Séminaire Contrôle EDP, Département de Mathématiques d'Orsay, 14 avril 2005.

Activités de recherche
Research activities

Thematic presentation of personal and supervised research

"La façon dont on trouve n'est pas celle dont on prouve."

Albert Einstein, physicien allemand, 1879-1955.

My research activity, so far, has followed three major paths :

- extension of my PhD work to the stability of systems described by partial differential equations in infinite dimension, applied to the Multi-Models approach,
- a new topic on the stability and the control of processes with a moving interface,
- extension of my postdoctoral research on natural invariants of the systems using a new approach : the port Hamiltonian systems written with the Riemann invariants formalism.

The research work in these themes has been disseminated at national and international levels through regular publications, students lectures, seminars and conferences ; and has also been part of the research pursuits of the doctoral candidates under my supervision :

- on the topic of Linear Operator Inequalities, **3 master PhD**, 2 international peer-reviewed journal papers (**TCST-2013, AMCS-2012**), 3 international conferences (**ECC-2013, 18thIFAC-2011, CIFA-2010**) and 2 invited lectures (**SAR-2012, EDP-2012**).
- on the extrusion process, **1 PhD student**, 1 peer-reviewed journal paper (**JESA-2011**), 4 international conferences (**CPDE-2013, CIFA-2012, CDC-2011,CDPS-2009**) and 1 invited lecture (**INSA-2009**).
- on the port Hamiltonian approach, **1 master student**, **1 collaboration with a PhD student**, 2 international peer-reviewed journal papers (**IEEE-TAC-2009, NHM-2008**) and 5 international conferences (**MTNS-2014, NOLCOS-2010, ECC-2009, CIFA-2008-1** &**-2**) and 1 invited conference, (**Irr-Wkshp-2008**).

Others publications, related mostly to my Phd and post-doctoral research, are not referenced here, but listed in the publications list.

CHAPITRE 4

Stability by linear operators inequalities

"Tout changement est une menace pour la stabilité."
Le Meilleur des mondes

Aldous Huxley, Romancier et essayiste britannique (1894-1963).

Sommaire

4.1 Introduction

These works were actually initiated during my PhD research, when I was conducting experiments on the benchmark located at ESISAR school, a benchmark developed for an irrigation channel.
"What happens if you are not around an equilibrium profil ?" Eduardo Mendes asked me, the professor who had received me in his lab.
What a good question ! ! "Let's try it"...
And we make our first attempt on Multi-Models, experimentally and successfully.
Since that day, my question was how to prove it. Back in my lab, the LVR, I did some simulations, but at that time I had no idea about how to solve this problem from a theoretical point of view.
Some years later, I met Mickael Rodrigues, assistant professor at LAGEP lab's, who specializes among various other fields in the Multi-Models theory. Our discussions led to a strong collaboration in 2007, and two years later we developed this new topic inside the laboratory.

4.2 Statements

Regulation of irrigation channels has received an increasing interest over the last three decades. Water losses in open channels are very large due to inefficient management and control. To avoid overflows and to satisfy the water demand, the level of instrumentation (e.g., operating motor-driven gates, water level measurements) and automation in open channel networks increase [56]. In order to deliver water, it is important to ensure that the water level and the flow rate in open channels remain at certain values. The difficulty of this regulation problem is that only the gate positions can meet performance specifications. Such problems can be solved by designing boundary control laws in order to satisfy the control objectives : to maintain water level or flow rate at given values.

The open surface channels couple transport phenomena and delay phenomena, so they have complex nonlinear dynamics. The dynamic of such distributed parameters systems can be represented by hyperbolic Partial Differential Equations (PDE) : the equations of de Saint-Venant or shallow water equations, which depend on time and space [55, 63, 80]. Some studies take into account the uncertainties and apply robust control approaches [51, 50]. Studying the nonlinear dynamics directly is also possible as in [22, 31, 50, 82]. The Riemann approach has also been used to prove stability results for systems of two conservation laws [36] and for systems of larger dimensions in [49]. Recently, it has been also coupled with LMI [7]. The Lyapunov techniques have been used in [12, 21, 22].

In practice, industrial processes such as mining, chemical or water treatment processes are complex systems, characterized by by multiple operating regimes. Multi-Models methods split the operating range of a system into separate regions where local models describe each region [60] for control and Fault Diagnosis purposes [8, 34, 66]. Each local model is defined as a Linear Time Invariant (LTI) model for each operating point. The Multi-Models philosophy is based on weighting functions which ensure the transition between the different local models. Some authors have studied gain scheduling strategy for example in [48] or Linear Parameter Varying (LPV) controllers [68].

The use of Multi-Models representation for the study of the stability of a system described by nonlinear PDE has been examined in [7, 24, 26]. The nonlinear PDE stability is studied by extending the common approaches based on finite dimension to infinite dimension.

Initially, our collaboration with M. Rodrigues began in 2007 with some discussions and the supervision of a master, S. Chabou, which allowed to identify the main difficulties. With a second master's student, M. Diagne, we went further and a theoretical proof developed for the closed loop stability of the infinite dimensional system under an Integral [18] and a Proportional Integral (PI) controller with identical gains [24, 26]. These restrictions were due to the complexity of proving the stability for the systems under infinite dimensional constraints. A particular control structure is used, the internal model boundary control (IMBC), because of its natural property of predictive control that is well adapted here to systems with delays.

The third master student, Y. Wu, proposed an approach in finite dimension [27] which led us to analyze of the closed loop stability of the de Saint-Venant PDE in infinite dimension with a general PI, using Multi-Models and the Internal Model Boundary Control (IMBC) structure [29]. A variable elimination technique, similar to finite-dimensional systems [9, 67, 70], was used in order to solve a BOI (well known as Bilinear Matrix Inequality (BMI) in finite dimension) problem by the resolution of two LOI (well known as Linear Matrix Inequalities (LMI) in finite dimension).

4.3 Problem statement about channel regulation

The control problem concerns the stabilization of the water flow rate and/or the water height around an equilibrium for a reach denoted by $(Z_e(x), Q_e(x))$.

4.3.1 A model of a reach

The channel is supposed to have a sufficient length L (from $x = 0$ to $x = L$) such that one can consider that the lateral movement is uniform. The water flow rate $Q(x,t)$ and the height $Z(x,t)$ of the water are the state variables. The nonlinear PDE of de Saint-Venant, which describe the flow on the channel, are [16, 35] :

$$\begin{cases} \partial_t Z = -\partial_x \frac{Q}{b}, \\ \partial_t Q = -\partial_x (\frac{Q^2}{bZ} + \frac{1}{2} g b Z^2) + gbZ(I - J), \end{cases} \tag{4.1}$$

$$y(t) = C[Z(x,t) \;\; Q(x,t)]^T \tag{4.2}$$

$$Z_0(x) = Z(x,0), Q_0(x) = Q(x,0) \tag{4.3}$$

$\forall x \in \Omega = (x_{up}, x_{do}) = (0, L)$, $t > 0$, $C : (L^2(0,L))^2 \to \mathbb{R}$. I is the slope, b is the channel width, g is the gravity constant. J is the friction slope from the formula of Manning-Strickler and R is the hydraulic radius. The considered boundary conditions $\forall x \in \Gamma = \partial\Omega$ are two underflow gates. The controlled variable is defined as follows :

$$Q(x,t) = U(t)\Psi(Z(x,t)) \tag{4.4}$$

with $\Psi(Z) = \kappa\sqrt{2g(Z_{up} - Z_{do})}$. Z_{up} is the water height at upstream of the gate, Z_{do} is the water height at downstream of the gate, κ is the product of the channel width and the water flow rate coefficient of the gate. The gate opening $U(t)$ is the control at upstream ($U_{up} = U_0$) and at downstream ($U_{do} = U_L$). The output variable is the downstream water level, i.e. $Z(L)$.

4.3.2 A regulation model

The fluvial case, i.e. the subcritical case [55], is considered. Let $\xi(t) = (z(t) \;\; q(t))^T$ be the linearized state variable, then the model around the equilibrium state $(Z_e(x) \;\; Q_e(x))^T$ is :

$$\begin{aligned} \partial_t \xi(x,t) &= \mathcal{A}\xi(x,t) = A_1(x)\partial_x \xi(x,t) + A_2(x)\xi(x,t), && (4.5) \\ F_b \xi(t) &= B_b u(t) \text{ and } \xi(0,t) = \xi_0(t), && (4.6) \end{aligned}$$

where A_1 and A_2 are matrices of the space variable x. The linearized boundary conditions (4.6) are equivalent to :

$$q(x_{up},t) = U_{up,e}\partial_z \Psi(Z_e(x_{up},t))z(x_{up},t) + u_{up}(t)\Psi(Z_e(x_{up},t)), \quad (4.7)$$
$$q(x_{do},t) = U_{do,e}\partial_z \Psi(Z_e(x_{do},t))z(x_{do},t) + u_{do}(t)\Psi(Z_e(x_{do},t)), \quad (4.8)$$

where $U_{up,e}$ and $U_{do,e}$ are the upstream and downstream gate openings (respectively) at the equilibrium and $u_{up}(t)$, $u_{do}(t)$ are the variations of these opening gates to be controlled.

The control problem is to find the variations of $u_{up}(t)$ at extremity $x = x_{up}$ and $u_{do}(t)$ at extremity $x = x_{do}$ of the reach such that the downstream water level, $Z(x_{do},t) = Z(L,t)$ (measured variable) tracks a reference signal $r(t)$. The reference signal $r(t)$ is chosen for all cases either constant or non-persistent (a stable step answer of a non-oscillatory system).

As it is a delay system, the control scheme based on the Internal Model Boundary Control (IMBC) is adopted [20]. This control strategy integrates the process model in real time and allows to regulate the water height in all the points of the channel by taking into account the error between the linearized model and the real system (or the nonlinear model for the simulations).

4.3.3 Open-loop system stability

Equation (4.5) describes the open loop system dynamics. In this representation, the state vector $\xi(x,t)$ is not explicitly linked with the boundary control. In order to design an output feedback and to study the closed-loop stability, a distribution operator D of control at the boundary is introduced [32][1], $D : \mathcal{C}^k([0,\infty], \mathbb{R}^n) \rightarrow (L^2(0,L))^2$. It is a bounded operator such that $Im(D) = Ker(\mathcal{A})$ and $Du \in D(\mathcal{A})$ and [20, 32, 71] :

$$\xi(x,t) = \varphi(x,t) + Du(t). \quad (4.9)$$

This operator is naturally null in the domain of $\mathcal{A}(x)$ as it is active only on the boundary of the domain. This change of variables allows to get a Kalman representation of the system [2, 32, 71] :

$$\begin{aligned} \partial_t \varphi(x,t) &= \mathcal{A}(x)\varphi(x,t) - D\dot{u}, && (4.10) \\ \varphi(x,0) &= \varphi_0(x) = \xi_0(x) - Du(0), && (4.11) \\ y(t) &= C\varphi(x,t) + CDu(t). && (4.12) \end{aligned}$$

It has been proved that the open loop system (4.10-4.12) is exponentially stable [20], as the operator of the linearized system in infinite dimension generates an exponentially

1. Regularity coefficient is generally taken as $k = 2$.

stable C_0-semigroup. Moreover, under a PI-control $u(t) = \alpha_i K_i \int \varepsilon(s)ds + \alpha_p K_p \varepsilon(t) \in U = \mathbb{R}^n$, $u \in C^k([0,\infty], U)$, the stability of the closed-loop nonlinear system is ensured under some specific conditions on the gain synthesis. They are deduced from the properties of the IMBC structure and from the stability of the closed-loop linearized system.

For example, for the tuned gains of the PI-control the stability conditions are ensured if :

$$0 \leqslant \alpha_i < \alpha_{i,max} = \min_{\lambda \in \Gamma} (a \| R(\lambda; \mathcal{A}_e) \| + 1)^{-1}, \tag{4.13}$$

$$0 \leqslant \alpha_p < \alpha_{p,max} = (\sup_{\lambda \in \Gamma} a \| R(\lambda; \mathcal{A}) \|)^{-1}, \tag{4.14}$$

where $\mathcal{A}_e$ is a part of the series development of the closed loop operator [20], $R(\lambda; K)$ is the resolvent operator of K, and a is a constant which depends on $\mathcal{A}_e$.

These theoretical results have been corroborated by simulations and experimentation [20]. These experiments have shown the limitations due to the linearization around an equilibrium state. My first attempt with a Multi-Models approach was successfull [20]. However, it was not optimal and no theoretical proof was given. The first approach by an integral control [26] had been extended to a PI control in [24], but the proportional and the integral gains were equal.

In this chapter, I present the final results obtained in collaboration with M. Rodrigues, which extend the previous results in infinite dimension to the case where the proportional gain is different from the integral gain. The theoretical proof in finite dimension with $K_{int} \neq K_{pr}$ [67] is extended to infinite dimensional systems. In order to control the water level over a wide operating range, a set of models is considered around judicious operating regimes : a control is synthesized and activated on the intervals when the system gets through the intervals.

4.3.4 A Multi-Models representation of de Saint-Venant's Equation

The Multi-Models representation [67, 26] of de Saint-Venant's PDE around N operating points is defined by the following equations :

$$\begin{aligned} \partial_t \xi(x,t) &= \sum_{i=1}^{N} \mu_i(\zeta(t)) \mathcal{A}_i(x) \xi(x,t) \text{ with } \mathcal{A}_i(x) = A_{1,i}(x)\partial_x + A_{2,i}(x), \\ \xi_0(x) &= \xi(x,0), \end{aligned} \tag{4.15}$$

where $\mathcal{A}_i(x)$ is the operator which corresponds to the i^{th} equilibrium state. $\zeta(t)$ is a function which depends on some decision variables directly linked with the measurable state variables and eventually to the input. $\mu_i(\zeta(t))$ are the weighting functions which activate the control law in function of the output of the process Z_L. They belong to a convex set such that

$$\sum_{i}^{N} \mu_i(\zeta(t)) = 1 \text{ and } \mu_i(\zeta(t)) \geq 0.$$

In the following section, the control law synthesis by LOI techniques is considered.

4.4 Study of the closed-loop system stability by LOI

In this part, the closed loop structure is studied under a proportional integral feedback. Let us recall that the aim is to control the water height over all the operating range, so that the output $y(t)$ is not the variations around an equilibrium but the total water height. To this end, the output $y(t)$ is modified :

$$y(t) = C\xi(x,t) + Eq(x,t),$$

where $Eq(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t))(Z_{e,i}(x,t)\ Q_{e,i})^T$ is the equilibrium state and for this paper $CEq(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t))Z_{e,i}(L)$ as the aim is to regulate the water level at $x = L$.

4.4.1 Closed-loop structure for a proportional integral feedback

Let K_{int} and K_{pr} be the integral and proportional gains respectively. It follows that [20] :

$$u(t) = K_{int} \int [r(\tau) - y(\tau)]d\tau + K_{pr}[r(t) - y(t)], \tag{4.16}$$

where $r(t)$ is the physical water level wanted (not the variations). So by using (4.9) we get :

$$y(t) = C\varphi(x,t) + CEq(x,t) + CDu(t), \tag{4.17}$$

and by replacing $y(t)$ into the equation (4.16), it becomes :

$$\begin{aligned} u(t) \quad = \quad & K_{int} \int [r(\tau) - (C\varphi(x,\tau) + CEq(x,\tau) + CDu(\tau))]d\tau \\ & + K_{pr}[r(t) - (C\varphi(x,t) + CEq(x,t) + CDu(t))] \end{aligned} \tag{4.18}$$

In each local model, $Eq(x,t)$ is a piecewise function ($\dot{E}q(x,t) = 0$) 2. This is also the case of $r(t)$. So, $\dot{u}$ can be simplified to :

$$\dot{u}(t) = K_{int}[r(t) - C\varphi(x,t) - CEq(x,t) - CDu(t)] - K_{pr}C\mathcal{A}_i(x)\varphi(x,t) \tag{4.19}$$

By replacing $\dot{u}$ into the equation (4.10) and with $\tilde{K}_{int} = D\ K_{int}$, $\tilde{K}_{pr} = D\ K_{pr}$, the expression of the closed-loop system can be expressed as follows :

$$\begin{aligned} \partial_t\varphi(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t))\big[(\mathcal{A}_i(x) + \tilde{K}_{int}C + \tilde{K}_{pr}C\mathcal{A}_i(x))\varphi(x,t) \\ + \tilde{K}_{int}\big(CDu(t) + CEq(x,t) - r(t)\big)\big] = \sum_{i=1}^{N} \mathcal{M}_i(x,t). \end{aligned} \tag{4.20}$$

The stability conditions are ensured by using a quadratic Lyapunov function [68] guaranteeing the convergence of the water height to the reference $r(t)$ over the widest operating range.

4.4.2 Lyapunov stability analysis

Let us consider :

$$V(\varphi(x,t),t) = \langle \varphi(x,t), P\varphi(x,t) \rangle, \tag{4.21}$$

2. The following notation is considered : $\partial_t\phi = \dot{\phi}$ whatever the function ϕ.

where $\langle .,. \rangle$ is the considered inner product. The Multi-Models representation of the linearized PDE of de Saint-Venant defined by equation (4.20) is asymptotically stable if there exists an operator $P > 0$, such that :

$$\langle \dot{\varphi}, P\varphi \rangle + \langle \varphi, P\dot{\varphi} \rangle = -\langle \varphi, \varphi \rangle. \tag{4.22}$$

The main difference here between the stability result in finite and infinite dimension [26], lies in the *inequality* of the Lyapunov function for finite dimensional systems and *equality* for infinite ones (4.22). This equality complexity can be removed in some cases ; as for example for operators with compact resolvent [14, 20, 72] or [69]. In this case, the same inequality from finite dimension is a sufficient and necessary condition for the infinite dimensional case ; it needs to satisfy the spectral growth assumption [72, 32]. Moreover, for the equations of de Saint-Venant, it has been shown that the operator has a compact resolvent [20] so it satisfies the spectral growth assumption. Then, by taking into account (4.20)-(4.22), one has to prove the following inequality :

$$\langle \mathcal{M}_i, P\varphi \rangle + \langle \varphi, P\mathcal{M}_i \rangle \ < 0, \tag{4.23}$$

where $\mathcal{M}_i$ is defined in (4.20).
The development of the inequality (4.23) leads us to consider an inequality for each local system of index i such that :

$$\begin{aligned} &\langle [\mathcal{A}_i + \tilde{K}_{int}C + \tilde{K}_{pr}C\mathcal{A}_i]\varphi(x,t), P\varphi(x,t) \rangle \\ &+\langle \tilde{K}_{int}[CDu(t) + CEq(x,t) - r(t)], P\varphi(x,t) \rangle \\ &+\langle \varphi(x,t), P[\mathcal{A}_i + \tilde{K}_{int}C + \tilde{K}_{pr}C\mathcal{A}_i]\varphi(x,t) \rangle \\ &+\langle \varphi(x,t), P\tilde{K}_{int}[CDu(t) + CEq(x,t) - r(t)] \rangle \ < 0. \end{aligned} \tag{4.24}$$

In the inequality (4.24), which defines the stability condition of the system $\forall i$, the control parameter u appears ; this is a difficulty for the gain synthesis : $\tilde{K}_{int}$, $\tilde{K}_{pr}$.

A first approach was made in [24] with $K_{pr} = K_{int}$. The previous results were improved as K_{int} is considered different from K_{pr}. It has been proved that a good choice of K_{int} and K_{pr} based on semigroup theory is $K_{int} = -\alpha_i[CD]^\dagger$ and $K_{pr} = \alpha_p[CD]^\dagger$ (where † stands for the right pseudo-inverse) in [20]. α_i and α_p are defined in (4.13)-(4.14). So, it can be assumed that $\exists \beta \in \mathbb{R}$ such that $K_{pr} = \beta K_{int}$, i.e. $\tilde{K}_{pr} = \beta \tilde{K}_{int}$. Then, the equation (4.24) becomes :

$$\begin{aligned} &\langle [\mathcal{A}_i + \tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i]\varphi(x,t), P\varphi(x,t) \rangle \\ &+\langle \tilde{K}_{int}[CDu(t) + CEq(x,t) - r(t)], P\varphi(x,t) \rangle \\ &+\langle \varphi(x,t), P[\mathcal{A}_i + \tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i]\varphi(x,t) \rangle \\ &+\langle \varphi(x,t), P\tilde{K}_{int}[CDu(t) + CEq(x,t) - r(t)] \rangle < 0 \end{aligned} \tag{4.25}$$

Note that the open-loop system 4.5)-(4.8) is exponentially stable as the closed loop one under a PI-control, with gains correctly tuned [20] for a time t well chosen. So, one can assume that $\exists k > 0$, such that :

$$\mid C\xi(x,t) + CEq(x,t) - r(t) \mid \leq k \mid C\varphi(x,t) \mid \tag{4.26}$$

and with $\alpha = (k+1)\varepsilon_{(\varphi^T P \tilde{K}_{int} C\varphi)}$ [26] :

$$\langle \varphi, P\tilde{K}_{int}(CDu(x,t) + CEq(x,t) - r(t)) \rangle \ \leq \ \langle \varphi, \alpha P\tilde{K}_{int}C\varphi \rangle \tag{4.27}$$

For finite dimensional systems, a stability study has been given in our paper [67], by using well-known linear techniques, but not developed for infinite dimensional systems. The main contribution consists in a tool developed using the semigroup theory.

Proposition 4.4.1. *Let Z be a Hilbert space and let G, U, V, X four linear operators on Z such that $G : D(G) \subset Z \to D(G)$. The domains of U, V and X are densely defined on $D(G)$, the domain of G. X is a self-adjoint operator such that $\|X\| \leq 2\sigma$.*
If $\exists \sigma \in \mathbb{R}\backslash\{0\}$ which satisfy, $\forall \phi,\ \varphi \in D(G)$

$$\langle G\varphi + U^*\phi, \varphi\rangle + \langle U\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{4.28a}$$

$$\langle G\varphi + V^*\phi, \varphi\rangle + \langle V\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{4.28b}$$

then, the following inequality is also satisfied :

$$\langle G\varphi, \varphi\rangle + \langle U^*XV\varphi, \varphi\rangle + \langle V^*XU\varphi, \varphi\rangle \ < 0 \tag{4.40}$$

■

Proof:
See [29]. □

The following proposition extends our results [67] to infinite dimensional systems.

Proposition 4.4.2. *If there exist a self-adjoint operator P, matrices W_{int} and W_{pr}, scalars $\sigma, \gamma \in \mathbb{R}$, such that the inequalities (4.28) are satisfied with $G = \mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*$, $U^* = W_{int}$, $V = C\mathcal{A}_i$, then one gets the following inequalities :*

$$\langle(\mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*)\,\varphi + W_{int}\phi, \varphi\rangle + \langle W_{int}^*\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{4.41a}$$

$$\langle(\mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*)\,\varphi + (C\mathcal{A}_i)^*\,\phi, \varphi\rangle + \langle C\mathcal{A}_i\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{4.41b}$$

and the closed-loop system (4.20) under the PI control law (4.16) is stable. ■

Proof:
Let us consider (4.21) with the closed-loop system (4.20) under a PI control law (4.16). By considering the inequality (4.23), we can obtain (4.25) with $K_{pr} = \beta K_{int}$. Now, let us assume that the inequality (4.27) holds true, then (4.25) becomes :

$$\begin{aligned}\langle\Big(P[\mathcal{A}_i + \gamma\tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i]\Big)\,\varphi, \varphi\rangle& \\ +\langle\varphi, \Big(P[\mathcal{A}_i + \gamma\tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i]\Big)\,\varphi\rangle &< 0\end{aligned} \tag{4.42}$$

with $\gamma = 1 + \alpha$, $\tilde{K}_{int} = P^{-1}W_{int}$, $\tilde{K}_{pr} = \beta\tilde{K}_{int}$. The inequality (4.43) has two variables : β and W_{int} that lead to a BOI problem. By using proposition 4.4.1, the BOI problem can be solved as two LOI problem. The inequality (4.43) is equivalent to (4.40) with $G = \mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*$, $U^* = W_{int}$, $V = C\mathcal{A}_i$, $X = \beta Id$. Proposition 4.4.1 allows to conclude that if the inequalities (4.41) are satisfied, the Lyapunov inequality is also true. Thus the system (4.15) under PI control is stable. □

Remark 4.4.1. The inequalities (4.41) seem to be linked with the stability of each submodel in infinite dimension, with a Lyapunov Input-to-state stability (ISS) function [59, 15].

The aim of the section 4.5 is to show the simulated curves obtained with this method and the ones obtained on the experimental benchmark [20], and the simulations with the gains $\tilde{K}_{int} = \tilde{K}_{pr}$ [24, 26].

4.5 Simulation results

Two benchmarks are used for the simulations : the micro-channel of Valence (France) and the channel of Gignac (France). The simulations are based on a Chang and Cooper scheme [22, 26].

For both applications, the weighting function $\mu_i(\zeta(t))$ is first equal to 1 if the output's height is included into the validity domain of the model and 0 in the other case for each operating state. In a second time, the weighting functions are chosen as smooth functions and not piecewise constants.

The output of the system is the decision variable. The parameter $\zeta(t)$ is a function of this variable. Both coefficients β and σ of the proposition 4.4.1 are negative in both simulated cases, so the condition $\|X\| \leq -2\sigma$ is always satisfied whatever $X = \beta Id$. Here, the inequalities (4.41) are solved after discretization by LMI as in [67].

4.5.1 The micro-channel of Valence

An experimental validation has been performed on the Valence micro-channel, shown in figures (4.2) and (4.1). This pilot channel is located at ESISAR [3] /INPG [4] engineering school in Valence (France). It is operated under the responsibility of the LCIS [5] laboratory. This experimental channel (total length=8 meters) has an adjustable slope and a rectangular cross-section (width=0.1 meter). The channel is ended on the downstream by a variable overflow spillway and equipped with three underflow control gates (figures (4.2) and (4.1)). Ultrasound sensors provide water level measurements at different locations of the channel (figure (4.2)). Note that water flow is deduced from the gate equations and has not been measured directly.

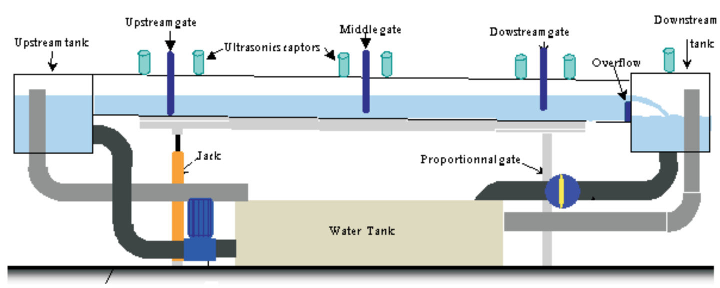


Figure 4.1 : *Pilot channel of Valence*

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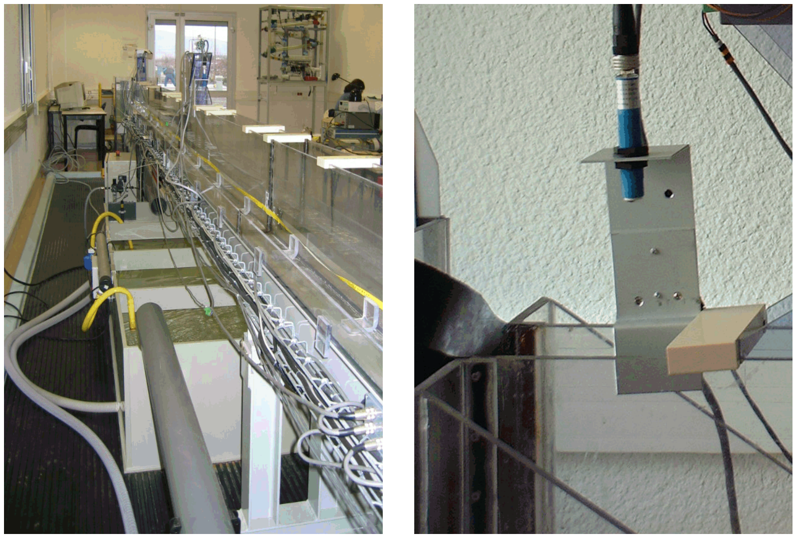

Figure 4.2 : *Pilot channel of Valence : gate and ultrasound sensors*

The equilibrium profiles have been chosen such that the calculated control law from the local models can be efficient over all the operating range of the water height [20].

Notice that it has been experimentally verified that a local model is valid around $\pm 20\%$ of an equilibrium profile. In order to assign references which are included between $0.06\ m$ and $0.2\ m$, the operating points at $x = 0$ are given in the Table 4.1.

Tableau 4.1 : *Initial set points for the simulation of the channels of Valence and Gignac*

	$z_{e1}(x=0)$	$z_{e2}(x=0)$	$z_{e3}(x=0)$
Valence	$0.06\ m$	$0.1\ m$	$0.16\ m$
Gignac	$1.09\ m$	$1.46\ m$	$1.68\ m$

The Figure (4.3) represents the dynamic evolution of the gates opening of the simulated system.

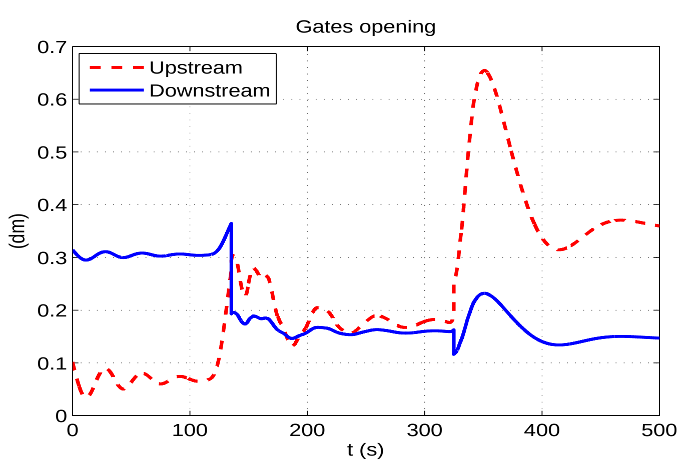

Figure 4.3 : *Valence channel simulation : Gates opening*

The following simulation (Figure (4.4)) compares PI controllers : one with gains $K_{int} = K_{pr}$ [24] and the new one with gains $K_{int} \neq K_{pr}$.

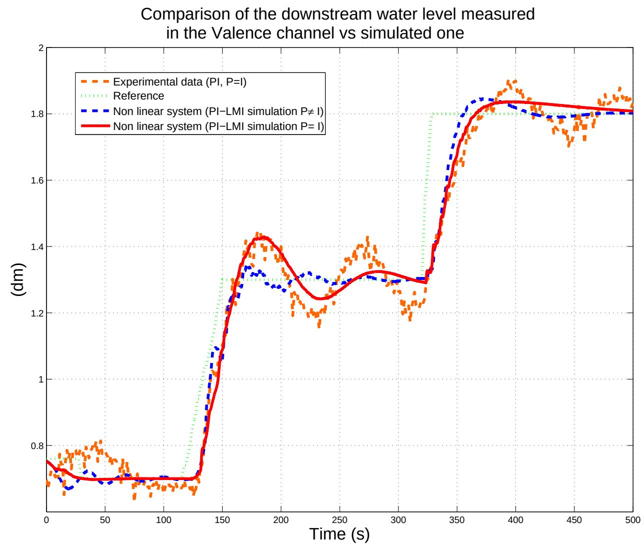

Figure 4.4 : *Valence channel simulation : Comparison of the downstream water level*

It can be observed in Figure (4.4) that the water height convergence with the new PI controller is better than the one obtained with $K_{int} = K_{pr}$ [24] and the overshoot is less too. It is also better than the experimental PI which has been implemented in [20].

The improvements here are achieved by the introduction of weighting functions μ_i taken different from $\{0, 1\}$.

From a theoretical point of view, the results were developed for any functions μ_i with the regularity needed, but numerically it was implemented initially only with constant functions. Recently, we improved the performance of the controller by the use of continuous functions.

It can be seen in figures (4.5), that around time instants $t = 250, 1000s$ and $2180s$, the new weighting functions $\mu_i \in [0, 1]$ (depicted in Fig (4.6)) outperform previous results obtained for the old weighting functions $\mu_i \in \{0, 1\}$ of paper [29]. Indeed, the water level at downstream tracks the reference better and avoids some overtaking. It can be remarked that the opening gates in Figure (4.6) get very similar dynamics in spite of different weighting functions μ_i : however, just around time instant $t = 900s$ and $t = 2000s$, the opening gates are little bit different which corresponds to the transition between the weighting functions. In [29] the weighting functions were considered only as a switch between models and the system model did not remain correct during this switching. The new weighting functions $\mu_i \in [0, 1]$, provide a real interpolation between the local models improving the dynamics of the nonlinear system at every time instant.

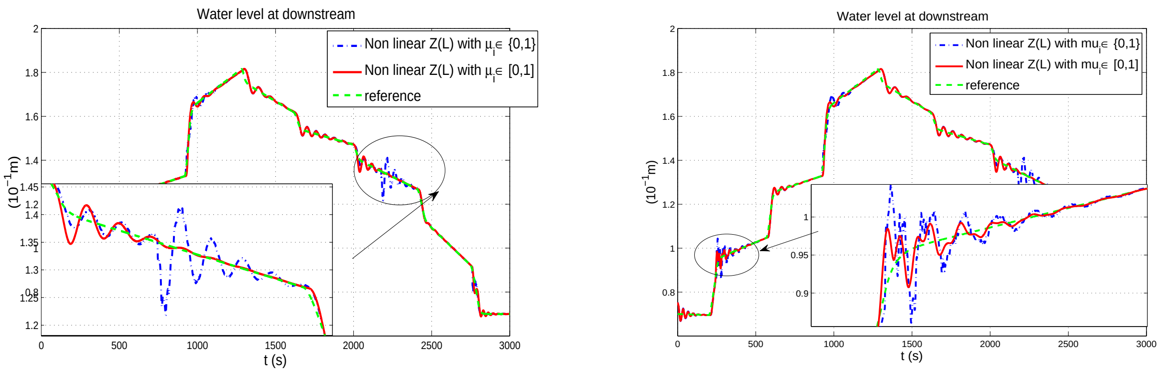

Figure 4.5 : *Valence channel simulation ; Comparison of the downstream water level*

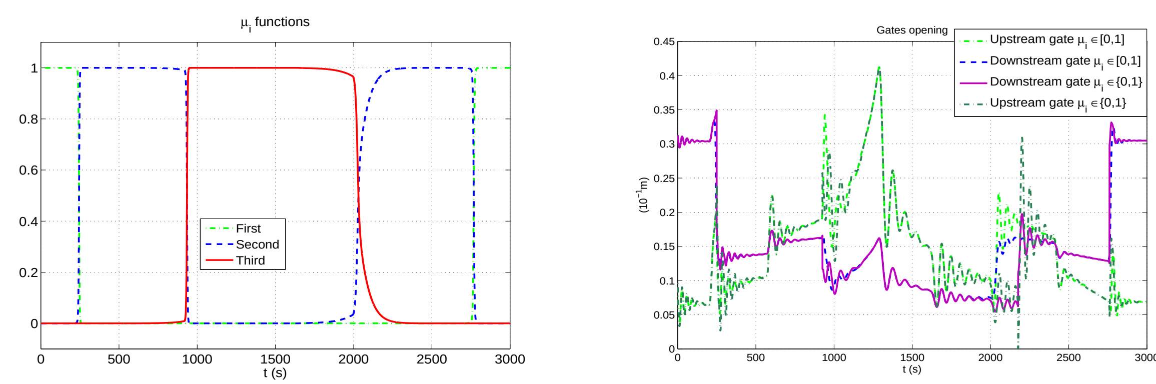

Figure 4.6 : *Valence channel simulation ; μ_i functions and gates opening*

Figure 4.7 : *Gignac channel*

4.5.2 The channel of Gignac

To get a better feedback, we have also studied a channel which is located in Gignac (France) (Figure (4.7)).

The following set of parameters of this channel is considered :

– $L = 2272\ m$ is the length of the channel,
– $b = 3\ m$ is the width of the channel,
– $N = 40$ is the number of the discretizated points,
– Z_L is the water height to regulate such that $1.7\ m < Z < 2.5\ m$.

The figure (4.8) shows that the output converges to the reference over a wide operating range.

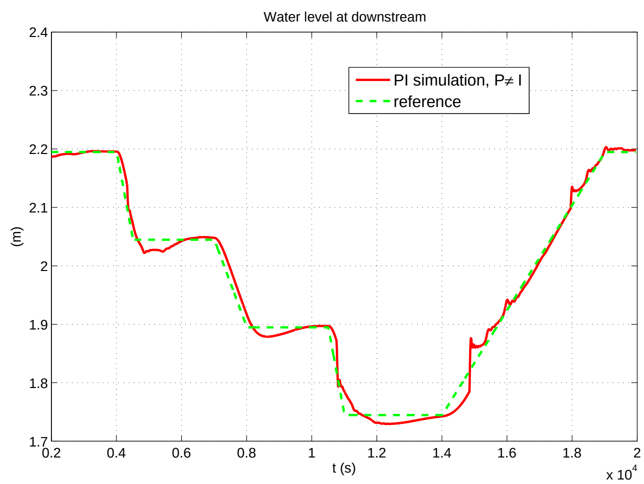

Figure 4.8 : *Gignac channel simulation : Comparison of the downstream water level*

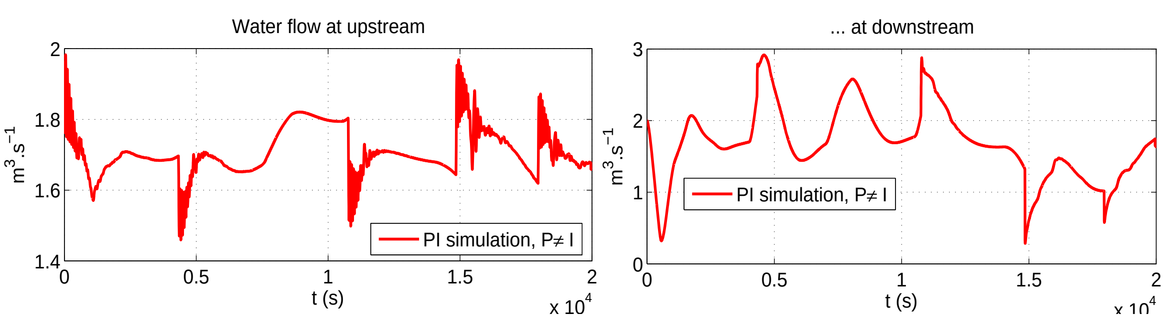

Figure 4.9 : *Gignac channel simulation : Water flows*

In Fig. (4.9), the upstream and downstream water flows stay in physical proportion, which is an important practical point. In Fig (4.12), it can be seen that the use of weighting functions $\mu_i \in [0, 1]$ (Fig (4.11)) allows to avoid some overtaking at time instants $t = 1200s$ and $t = 2800s$ that corresponds also to the transition of models. Again, the old weighting functions were only a switch that have been replaced in this paper by a real interpolation between the local models to better track the reference water level. Moreover, in Fig. (4.10), it can be seen that the use of new weighting functions allows to avoid big damping for the upstream water flow around the same time instant $t = 1200s$ and $t = 2800s$. The downstream water flow is also improved although the improvement is less important. The gates opening in Figure (4.11) at the time instant $t = 1200s$ and $t = 2800s$, are more linear without big variations.

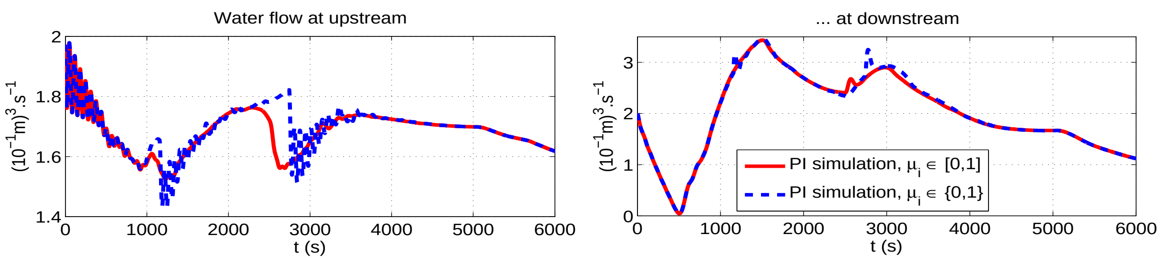

Figure 4.10 : *Gignac channel simulation ; Comparison of the water flow at upstream and downstream*

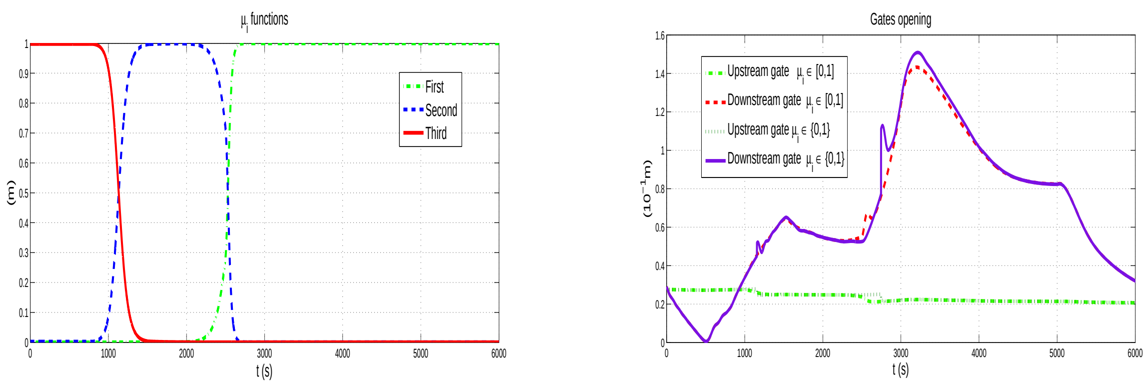

Figure 4.11 : *Gignac channel simulation ; μ_i functions and gates opening*

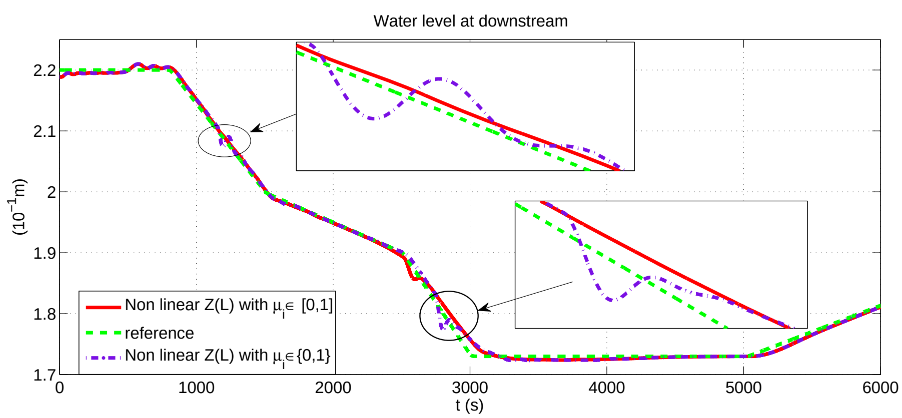

Figure 4.12 : *Gignac channel simulation ; Comparison of the downstream water level*

The last figure (4.14) presents another type of reference tracking for the downstream water level. It can be seen that by the use of the new weighting functions, all the overshoots have totally disappeared. Indeed, at time instant $t = 800s, 4000s, 5800s, 8600s$ which correspond to the transitions between local models, the new weighting functions contribute to get a smoother water level and better tracking. It can be highlighted that the transitions in (4.13) cover between $500s$ to $1500s$ i.e. the Multi-Models strategy takes all its sense by the fact that the use of a single model during these transitions are not judicious. The weighting of local models during these transitions allows to get better performance from a practical point of view by avoiding overtaking and bad behavior of opening gates as presented in Figure (4.13). We can see that around $t = 800s$ and $t = 8600s$, the improvement is significant.

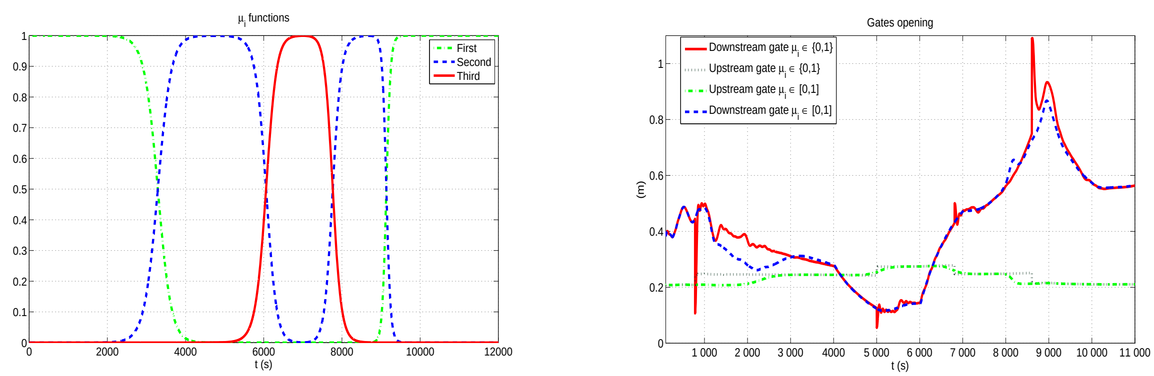

Figure 4.13 : *Gignac channel simulation ; μ_i functions and gates opening*

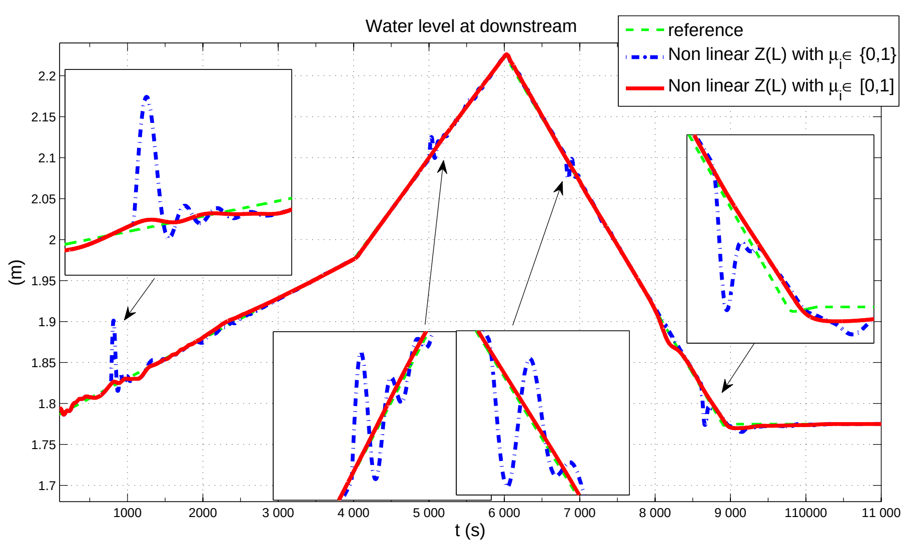

Figure 4.14 : *Gignac channel simulation ; Comparison of the downstream water level*

4.6 Conclusion

First attempts of a Multi-Models approach on irrigation channels control, through an IMBC structure, have been realized some years ago [20]. Good experimental results were obtained which showed promising results but a theoretical basis was lacking. The first theoretical results in order to design the feedback gain through LMI have been realized in the case of an Integral controller in [26]. Preliminary results of a PI controller in a particular case ($K_{int} = K_{pr}$) have been published in [24] for infinite dimensional systems and with $K_{int} \neq K_{pr}$ in [67] for finite dimensional systems. In the last paper, we took into account the more general case of PI controller with $K_{int} \neq K_{pr}$ for infinite dimensional systems. We synthesize the new PI controller feedback gains by solving a BOI problem. Initially the weighting functions were equals to 1 or 0. Recently, we improve the previous paper [29] by taking non-piecewise constant ($\mu_i(\zeta(t)) \in [0, 1]$). The last step should be to valid our approach through experimentation.

CHAPITRE 5

Stability and control of processes with a moving interface

"Les esprits mobiles ne sont pas garantis contre les idées fixes. "

Raymond d'Alost, Ecrivain humoristique français.

Sommaire

5.1 Introduction

Following his masters with me, we have with Mamadou Diagne, proposed a doctoral research topic on the theme of the team "Process dynamics and control of systems of conservation laws". The subject was "Control of a infinite dimensional non linear process with a moving boundary : application to extrusion processes". The research team "DYCOP" is part of the cluster project "Chemistry and Environment" and is particularly involved in the work package "Intensification Processes" on a research project on modeling reactive extrusion processes, in collaboration with Rhodia. This project follows an initial study conducted as part of a Contract Research Program (CPR) Materials in collaboration with the Laboratory of Plastics and Biomaterials (UMR CNRS 5627). Process intensification is a crucial issue for the chemical industry in the context of sustainable development.
These methods are characterized by a very complex geometry which hosts various reactions according to the different zones of the reactor, and the quality objectives of the products which are not easily measurable. These zones are moving according the flow and lead to a dynamic model consisting of distributed parameter systems with a moving boundary. The control of such systems with a moving interface has not been addressed much in the scientific community and currently their automatisation requires the development more suitable methods.

The first step was to develop a model, for describing an extrusion process. Then the aim was to study its natural stability or stabilization in order to implement a controller for the infinite dimensional closed loop system.

5.2 Statement

An extruder is made of a barrel with one or two Archimedean screws rotating inside. At the output, the extruder is equipped with a die from which the material is extruded in the process (Fig. (5.1)). It is controlled by the barrel temperature and the screw speed so as to ensure the desired properties at the die (moisture, density, etc... of the food or the polymer).

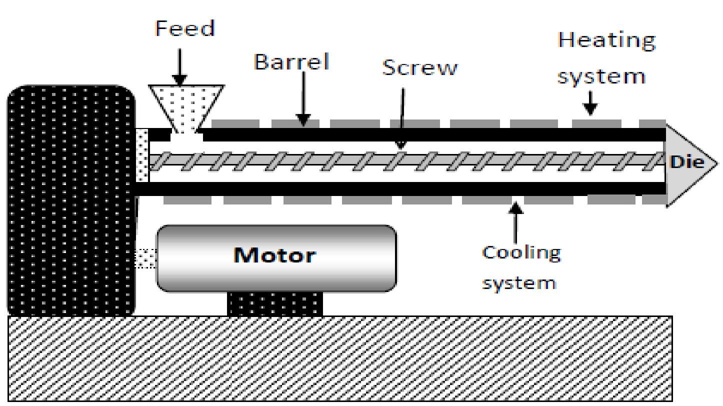

Figure 5.1 : *Description of the mechanism of an extruder*

The physical phenomena involved in the extrusion process consists of coupled non linear phenomena, such as viscous Newtonian or non Newtonian fluid flows, heat transfer and possible chemical reactions. The design of an extrusion process involves a complex modular geometry in function of the screw profile, allowing different mixing capacities along the extruder. The reader is referred to [75] for the steady-state modeling for design purpose and to [42, 43, 38, 39, 10] for dynamical models. For the control, a proportional integral PI feedback was developed in [44] and a multi-variable predictive control in [79].

First, we developed and analyzed an infinite-dimensional model : a simple 1 dimensional model consisting of two systems of conservation laws (with source terms) coupled with a moving interface. Then, the model was improved by the introduction of non constant viscosity and melt density. Indeed the melt density and the viscosity dynamics influence directly the balance equations and in particular the temperature evolution. Assumptions of constant viscosity and melt density stated initially to simplify the model construction are actually not realistic and the physical behaviour of the steady state was not physically realistic, as well as it can been observed on the figure (5.4).

5.3 The physical model

Following [45] and [52], the spatial domain of the extruder is split into two parts : the partially and fully filled zones according to the figure (5.2).
In the partially filled zone (PFZ) (or conveying zone), the pressure is supposed to be constant and equal to the atmospheric pressure P_0. In the fully filled zone (FFZ), the filling volume fraction (the filled volume fraction which may be related to the total mass density) is by definition equal to 1 and the resistance of the die generates a pressure gradient. The difference between the net forward flow at the die and the pumping capacity of the screws causes the displacement of the interface between the PFZ and FFZ.

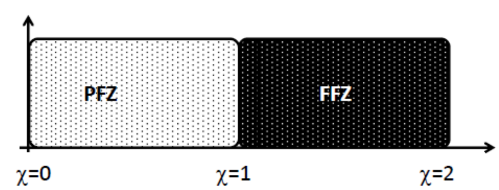

Figure 5.2 : *The 2-zones assumption in the extruder*

The dynamic model is derived from the mass and energy balances on a volume element for each zone. Some assumptions are made : the pitch of the screw $\xi(m)$ is uniform ; the flow is 1D and strictly convective, there exists a boundary between the PFZ and the FFZ corresponding to discontinuity of the filled volume (or filled volume fraction also called filling ratio f $(-)$) ; the extruded melt is composed of some species blended with water. Unlike in the previous works [19, 25], the melt density $\rho_0(kg\ m^{-3})$ and viscosity η $(Pa\ s^{-1})$ are supposed to be non constants and given by the WLF (William-Landel-Ferry) law [41] for example :

$$\eta(T) = \eta(T_0)a_T(T),\ log(a_T) = -\frac{C_1(T - T_0)}{C_2 + T - T_0} \tag{5.1}$$

where $\eta(T_0)$ is the viscosity at the initial temperature T_0. T is the temperature of the melt. C_1 and C_2 are constants depending up on the material. The melt density law is not given, and supposed to be non constant in the theoretical part, constant in the simulation part.

5.3.1 Model of the both Zones

Let l (m) be the position of the moving interface [17]. The mass balance equations in the PFZ, are written on the spatial domain $[0, l(t)]$, in terms of the filling ratio f_p and of the moisture content M_p $(-)$ [1] [45]. The energy balance is written in terms of the temperature T_p (K) of the mixture. The pressure P (Pa) is constant in the PFZ zone : $P(x,t) = P_0$.
In the FFZ zone, the model is reduced to the mass balance of water, written in terms of the moisture content M_f [2] and the energy balance is written in terms of the temperature T_f. The balances are written on the spatial domain $[l(t),\ L]$.

The pressure gradient in this zone is expressed as a function of the difference between the maximum flow and F_d :

$$\frac{\partial P(x,t)}{\partial x} = \eta_f \frac{V_{eff} N(t) \rho_0 - F_d}{B\rho_0} \tag{5.2}$$

$$\text{with } F_d = \frac{K_d}{\eta_f} \Delta P,\ \ \Delta P = (P(L,t) - P_0) \tag{5.3}$$

where

F $(kg\ s^{-1})$	Mass flow rate
F_d $(kg\ s^{-1})$	Net forward mass flow rate
$T_F(K)$	Barrel temperature
N $(rd\ s^{-1})$	Screw speed
K_d $(-)$	Geometric parameter
B	Geometric parameter
$S_{ech}(m^2)$	Exchange area between melt & barrel
$S_{eff}(m^2)$	Effective area
$V_{eff}(m^3)$ and $V_{eff} = \xi S_{eff}$	Effective volume
c_p $(J\ kg^{-1}\ K^{-1})$	Specific heat capacity
α $(Jm^{-2}s^{-1}K^{-1})$	Heat exchange coefficient
$\mu(J\ kg^{-1}\ K^{-1})$	Viscous heat generation parameter

Let us note that the gradient (5.2) is uniform.
The source term Ω_1 groups together the heat produced by the viscosity of the material and the heat exchange with the barrel. The heat transfer with the barrel and viscous dissipation created by the viscosity are defined in the term Ω_2 for the FFZ :

$$\begin{aligned}
\Omega_1 &= \frac{\mu_p \eta_p N^2(t)}{f_p(x,t)\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_p} - T_p) \\
\Omega_2 &= \frac{\mu_f \eta_f N^2(t)}{\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_f} - T_f)
\end{aligned}$$

1. Index p states for the PFZ zone
2. Index f states for the FFZ zone

All the functions depend on (x,t), for an easier reading they are omitted below. Let us define ϕ as [3] :

$$\phi = \begin{pmatrix} f_p & M_p & T_p & M_f & T_f \end{pmatrix}^T \tag{5.4}$$

then $$\frac{\partial}{\partial t}\phi = \begin{pmatrix} -\xi N I_3 & 0_{3\times 2} \\ 0_{2\times 3} & \frac{-F_d \xi}{\rho_0 V_{eff}} I_2 \end{pmatrix} \frac{\partial}{\partial x}\phi + \begin{pmatrix} 0 & 0 & \Omega_1 & 0 & \Omega_2 \end{pmatrix}^T \tag{5.5}$$

The boundary conditions at $x = 0$ are established under the assumption of continuity of the material flow rate, temperature, and flow momentum [25]. The mixing phenomena at the inlet are neglected :

$$\rho_0 N V_{eff} f_p(0,t) = F_{in}(t) \tag{5.6a}$$
$$T_p(0,t) = T_{in}(t) \tag{5.6b}$$
$$M_p(0,t) = M_{in}(t) \tag{5.6c}$$

where $M_{in}(t)$ and $T_{in}(t)$ are the moisture content and temperature of the matter at the inlet $x = 0$.

For the model of the moving interface $l(t)$, using [42, 43], we assume that the two zones are separated and l is defined by the discontinuity of the filling ratio f. In the PFZ, the filling ratio satisfies $f_p(x,t) < 1,\ x \in [0, l(t)[$ with $f_p(l^-,t) < 1$ and in the FFZ $f_f(x,t) = 1,\ x \in]l(t),\ L]$.
The dynamics of the moving boundary are obtained from the global mass balance on the FFZ zone :

$$\frac{dl(t)}{dt} = \frac{F(f_p(l^-,t)) - F_d}{\rho_0 S_{eff}(1 - f_p(l^-,t))} \tag{5.7}$$

At the interface $x = l(t)$, temperature and moisture content are supposed to be continuous :

$$T_p(l^-,t) \quad = \quad T_f(l^+,t),\ M_p(l^-,t) = M_f(l^+,t)$$

The third coupling relation between the two zones is given by the continuity of the momentum flux and allows to obtain the expression of the pressure at the die :

$$P(L,t) = P_0 + \frac{-[1 + \frac{K_d}{B}\int_{l^+}^{L} \rho^{-1} dx] + \sqrt{\Delta}}{\frac{2K_d^2}{\eta^2 \rho S_{eff}^2}} \tag{5.8}$$

$$\text{with } \Delta = \left[1 + \frac{K_d \int_{l^+}^{L} \rho^{-1} dx]}{B}\right]^2 + \overline{\Omega}_3\left(\overline{f}_p(1,t), N(t), l(t)\right)$$

$$\text{and } \overline{\Omega}_3 = \left(\frac{2K_d}{\eta S_{eff}}\right)^2 \left(\frac{V_{eff} N(t) \int_{l^+}^{L} \eta dx}{B\rho} + \xi^2 N^2(t) \overline{f}_p(1,t) - (1 - \overline{f}_p(1,t))\frac{P_0}{\rho}\right) \tag{5.9}$$

Remark 5.3.1. For polymers, the moisture variables do not exist. In the way they are coupled in the proposed model (5.5), both moisture equations for PFZ and the FFZ do not interfere with the others variables. So the following results which depend intrinsically of the temperature, the filling ratio and the moving boundary can be stated in a similar way without the moisture variables.

3. I_j stands for the identity matrix $j \times j$.

5.3.2 Linearized model

A classical change of the spatial variable was performed [17, 19] for the two zones in order to express the system in a fixed domain. It leads to two systems of conservation laws with source terms and, in addition a fictitious convection term due to the change of the spatial coordinate.

For each zone a change of the spatial coordinate is done. For the PFZ (for the FFZ respectively), the change from $[0, l(t)]$ (resp. $x \in (l(t), L)$) onto the interval $(0, 1)$ (resp. $(1, 2)$) is defined in this way :

$$\chi(x,t) = \frac{x}{l(t)}, \quad \text{respectively } \chi(x,t) = \frac{x + L - 2l(t)}{L - l(t)} \tag{5.10}$$

($x \in (0, L)$ becomes $\chi \in (0, 2)$ fig. (5.2)).

The PDE obtained are similar to (5.5) with the new state variable

$$\overline{\phi} = \begin{pmatrix} \overline{f}_p & \overline{M}_p & \overline{T}_p & \overline{M}_f & \overline{T}_f \end{pmatrix}^T$$

and $\overline{\Omega_i} = \Omega_i(\overline{\phi}), i = 1, 2$

(5.5) becomes :

$$\frac{\partial}{\partial t}\overline{\phi} = \begin{pmatrix} -\frac{1}{l(t)}\left[\xi N(t) - \chi \frac{dl(t)}{dt}\right] I_3 & 0_{3\times 2} \\ 0_{2\times 3} & -\frac{1}{L-l(t)}\left[\frac{F_d \xi}{\rho V_{eff}} - (2-\chi)\frac{dl(t)}{dt}\right] I_2 \end{pmatrix} \frac{\partial}{\partial \chi}\overline{\phi}$$
$$+ \begin{pmatrix} 0 & 0 & \overline{\Omega_1} & 0 & \overline{\Omega_2} \end{pmatrix}^T \tag{5.11}$$

With these new coordinates, the model equations include one fictive convective term, which depends on the velocity $\frac{dl(t)}{dt}$ of the interface [17] :

$$\frac{dl(t)}{dt} = \frac{F_d - \rho N(t) V_{eff} \overline{f}_p(1^-, t)}{\rho S_{eff}(1 - \overline{f}_p(1^-, t))} \tag{5.12}$$

The linearized system is then deduced for both zones and the state variable variation name is $\delta\overline{\phi}$.

Let us notice that all the linearized variables are denoted by δ.

The linearization is done around an equilibrium state deduced from $\partial_t \delta\overline{\phi_e} = 0$. The equilibrium variables are indexed by "e". For example T_e and $\Omega_{1,e}$ are the temperature and the heat transfer term at equilibrium, respectively.

The moving boundary $l(t)$ is fixed at the equilibrium and induces the following relation between the net flow F_{de} at the die and the screw rotational velocity N_e :

$$l_e = L - \frac{\frac{BP_0}{\rho_e \xi N S_{eff}} + \frac{B\eta_e f_{p,e}}{Kd} - \frac{B\xi N f_{p,e}(1-f_{p,e})}{S_{eff}}}{\frac{1}{\rho_e}\int_1^2 \eta d\chi - f_{p,e}\eta_e \int_1^2 \frac{1}{\rho} d\chi} \tag{5.13}$$

See [28] for more explanations.

The boundary conditions and the interface relations are easily deduced from their expressions in the initial variables (5.6) :

$$\begin{aligned} \delta\overline{f}_p(0,t) &= \delta\frac{F_{in}(t)}{\rho_0 N(t) V_{eff}} \\ \delta\overline{T}_p(0,t) &= \delta T_{in}(t) \\ \delta\overline{M}_p(0,t) &= \delta M_{in}(t) \end{aligned}$$

To get a unique model over the whole domain, the state variables are defined as follows in the domain $(0,\ 2)$. The restrictions on each zone PFZ and FFZ of those variables are equal to the value of the variables indexed by p and f respectively. For the moisture and the temperatures (inside and of the barrel), they are defined on $(0,\ 1) \cup (1,\ 2)$ by :

$$\begin{aligned}
\delta\overline{M}(\chi,t) &= \delta\overline{M}_p(\chi,t)\mathbf{1}_{(0,1)}(\chi) + \delta\overline{M}_f(\chi,t)\mathbf{1}_{(1,2)}(\chi)\\
\delta\overline{T}(\chi,t) &= \delta\overline{T}_p(\chi,t)\mathbf{1}_{(0,1)}(\chi) + \delta\overline{T}_f(\chi,t)\mathbf{1}_{(1,2)}(\chi)\\
\delta\overline{T}_F(\chi,t) &= \delta\overline{T}_{F_p}(\chi,t)\mathbf{1}_{(0,1)}(\chi) + \delta\overline{T}_{F_f}(\chi,t)\mathbf{1}_{(1,2)}(\chi)
\end{aligned}$$

where $\mathbf{1}_{(a,b)}(\chi) = 1$ when $\chi \in (a,b)$, else 0. The same is done for the viscosity and the melt density as they are functions of the temperature $\delta\overline{T}$. For the melt density, the linearization of (5.1) is given by

$$\delta\eta = y_\eta\delta(\overline{T}),\ \ y_\eta = -\eta(T_e)\frac{C_1C_2}{(C_2+T_e-T_0)^2} \tag{5.14}$$

The variation of the filling ratio is extended by 0 on the domain $(1,2)$, like the filling ratio is constant equal to 1 on it (the same is done for the position l of the interface with its variations $\delta l\,(t)$) :

$$\begin{aligned}
\delta\overline{f}_p(\chi,t) &= \delta\overline{f}_p(\chi,t)\mathbf{1}_{(0,1)}(\chi) + 0.\ \mathbf{1}_{(1,2)}(\chi)\\
\delta l(\chi,t) &= \delta l(t).\mathbf{1}_{(0,1)}(\chi) + 0.\ \mathbf{1}_{(1,2)}(\chi)
\end{aligned}$$

The space state is given by [25] :

$$X = \left(H^1(0,1)\times\mathcal{K}_{(1,\,2),0}\right)\times\left(H^1(0,2)\right)^2\times\left(\mathcal{K}_{(0,\,1)}\times\mathcal{K}_{(1,\,2),0}\right)$$

where $\mathcal{K}_{(1,\,2),0}$ is the set of null functions on the interval $(1,2)$ and $\mathcal{K}_{(0,\,1)}$ denotes the set of constant functions on $(0,1)$, which is isomorphic to $\mathbb{R}$. It can be used with a Hilbert space structure induced by the one of $H^1(0,1)\times\left(H^1(0,2)\right)^2\times\mathbb{R}$ and the norm used is noted by $\|\ \|$.

The following expressions give the linearized dynamics of the mobile interface :

$$\frac{d(\delta l)}{dt} = \alpha_l\delta l + \alpha_f\delta\overline{f}_p(1,t) + \alpha_N\delta N \tag{5.15}$$

and of the pressure at $x = L \leftrightarrow \chi = 2$

$$\delta\overline{P}(2,t) = \gamma_N\delta N + \gamma_f\delta\overline{f}_p(1,t) + \gamma_l\delta l \tag{5.16}$$

For more details, please see [17].
So the evolution of the linearized equations for the PFZ and FFZ zones, as well as the dynamics of the interface are given by, with the new state variable, notated with an abuse of notation still $\delta\overline{\phi}$

$$\delta\overline{\phi} = \left(\delta\overline{f}_p\ \ \delta\overline{M}\ \ \delta\overline{T}\ \ \delta l\right)^t$$

$$\frac{\partial}{\partial t}\left(\delta\overline{\phi}\right) = A\left(\delta\overline{\phi}\right) + B\begin{pmatrix}\delta N\\ \delta T_F\end{pmatrix} + \mathcal{C}\begin{pmatrix}\delta\eta\\ \delta\rho\end{pmatrix} \tag{5.17}$$

The operator A is decomposed as the sum of two operators : $A = A_1 + A_2$. The operator A_1 is the differential operator and is defined by :

$$A_1 = diag\left(\frac{-\xi N_e}{l_e}\mathbf{1}_{(0,1)}(\chi)\partial_\chi, \theta(\chi), \theta(\chi), \alpha_l\right) \tag{5.18}$$

where $\theta(\chi)$ deals with the transport equations on each zone :

$$\theta(\chi) = \frac{-\xi N_e}{l_e}\mathbf{1}_{(0,1)}(\chi)\frac{\partial}{\partial\chi} + \frac{-1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}\mathbf{1}_{(1,2)}(\chi)\frac{\partial}{\partial\chi}$$

The operator A_2 is defined by (all the coefficients are given in appendix) :

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{2,1} & 0 & A_{2,3} & A_{2,4} \\ \alpha_f\delta_{(1^-)}(\chi) & 0 & 0 & 0 \end{pmatrix} \tag{5.19}$$

$$\begin{aligned} \text{with : } A_{2,1} &= \left[(\beta_{p1,f} + \beta_{p4}\alpha_f)\mathbf{1}_{(0,1)} + \left(\beta_{f3}\gamma_f + \beta_{f5}\alpha_f\right)\mathbf{1}_{(1,2)}\right]\delta_{(1^-)} \\ A_{2,3} &= \beta_{p1,T}\mathbf{1}_{(0,1)}(\chi) + \beta_{f1,T}\mathbf{1}_{(1,2)}(\chi) \\ A_{2,4} &= (\beta_{p3} + \beta_{p4}\alpha_l)\,\mathbf{1}_{(0,1)}(\chi) + (\beta_{f3}\gamma_l + \beta_{f4} + \beta_{f5}\alpha_f)\,\mathbf{1}_{(1,2)}(\chi) \end{aligned}$$

where $\delta_{(1^-)}$ is given by $\delta_{(1^-)}\left(\delta\overline{f}_p(\chi,t)\right) = \delta\overline{f}_p(1^-, t)$.
The input operator is :

$$B = \begin{pmatrix} 0 & 0 & \beta_{2,N} & \alpha_N \\ 0 & 0 & \beta_{2,T} & 0 \end{pmatrix}^t \tag{5.20}$$

$$\begin{aligned} \text{where } \beta_{2,N} &= (\beta_{p2,N} + \beta_{p4}\alpha_N)\mathbf{1}_{(0,1)}(\chi) + (\beta_{f2,N} + \beta_{f3}\gamma_N + \beta_{f5}\alpha_N)\mathbf{1}_{(1,2)}(\chi) \\ \beta_{2,T} &= \beta_{p2,T}\mathbf{1}_{(0,1)}(\chi) + \beta_{f2,T}\mathbf{1}_{(1,2)}(\chi) \end{aligned} \tag{5.21}$$

The coefficients are defined by

$$\begin{aligned} \beta_{p1,f} &= -\frac{\mu\eta N_e^2}{\overline{f}_{pe}^2\rho_0 V_{eff}C},\ \beta_{p1,T} = -\frac{S_{ech}\alpha}{\rho_0 V_{eff}C}, \\ \beta_{p2,N} &= \frac{\mu\eta N_e}{\overline{f}_{pe}\rho_0 V_{eff}C} - \frac{S_{ech}\alpha(T_{F_{pe}} - \overline{T}_{pe})}{N_e\rho_0 V_{eff}C} \\ \beta_{p2,T} &= \frac{S_{ech}\alpha}{\rho_0 V_{eff}C},\ \beta_{p4} = \chi\frac{l_e}{\xi N_e}\beta_{p3} \\ \beta_{p3} &= \frac{1}{l_e}\left(\frac{\mu\eta N_e^2(t)}{\overline{f}_{pe}\rho_0 V_{eff}C} + \frac{S_{ech}\alpha(T_{F_{pe}} - \overline{T}_{pe})}{\rho_0 V_{eff}C}\right) \end{aligned}$$

and

$$\begin{aligned} \beta_{f1,T} &= -\frac{S_{ech}\alpha}{\rho_0 V_{eff}C} = -\beta_{f2,T},\ \beta_{f2,N} = \frac{2\mu\eta N_e}{\rho_0 V_{eff}C} \\ \beta_{f3} &= \frac{-K_d}{\rho_0\eta F_{de}V_{eff}C}\left(\mu\eta N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})\right) \\ \beta_{f4} &= \frac{-1}{(L-l_e)C\rho_0 V_{eff}}\left(\mu\eta N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})\right) \\ \beta_{f5} &= (2-\chi)\frac{1}{\xi F_{de}C}\left(\mu\eta N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})\right) \end{aligned}$$

The improvement is introduced by the third operator, $\mathcal{C}$ expressed independently. It is stemming from the variations of the melt density and the viscosity terms, initially supposed constant. Even if it does not modify the main operator A_1, it changes the temperature evolution in both zones of the extrusion process, via the operator A_2 :

$$\mathcal{C}(\chi) = \begin{pmatrix} 0 & 0 & \gamma_T & \frac{\gamma_l}{\eta} \\ 0 & 0 & \frac{\Omega_{1,e}}{\rho_e}\mathbf{1}_{(0,1)}(\chi) & \frac{\gamma_l}{\rho} \end{pmatrix}^t \tag{5.22}$$

with $\gamma_T = \frac{-\beta_{p1,f}}{\eta_e}\mathbf{1}_{(0,1)}(\chi) + \frac{\beta_{f2,N}+\beta_{f2,T}}{\eta_e}\mathbf{1}_{(1,2)}(\chi)$ and $\gamma_l = -\frac{\Omega_{3,e}}{\Delta_e} - \frac{K_d \Delta P}{\eta_e \rho_e S_{eff}(1-\overline{f}_{pe})}$ and $\eta_e = \eta(T_e)$.

Now, the question is : How this operator $\mathcal{C}$ is going to modify the open loop structure and properties, such as stability, and what are the consequence on the closed loop properties, and the previous control ?

For η and ρ constants, the well-posedness and the stability of the open loop around an equilibrium state have been proved in [19, 25] using the semigroup theory, the resolvent $R(\lambda, A)$ and the properties of $A = A_1\partial_x + A_2$.

5.4 PI control using an Internal Model Boundary Control (IMBC)

The control of an extrusion process is complicated, as the controllable variables are not the same all the time and depend on the extruded product. Its properties and parameters change from one product to another one. One choice could be to control the moisture and the temperature at the end of the fully full zone in order to get the properties desired after the die. As the model has been simplified, only the temperature is taken into account.
The control problem is to find the variations of the screw speed $N(t)$ and of the barrel temperature $T_F(x,t), \forall x \in [0,2]$ such that the temperature $T_f(L,t)$ tracks a reference signal $r(t)$. The reference signal $r(t)$ is chosen for all cases either constant or non-persistent.

5.4.1 Laws of η and ρ and open loop stability

The laws which describe the dynamic of the melt density and the viscosity are numerous and depend on the characteristics of the melt used ; food, polymers, etc... Usually, for food, the law of the evolution of the viscosity (and it is quite the same for the melt density) is a function of the temperature.

Here, it is proposed to define an empirical law for both variables, as follows :

Proposition 5.4.1. *If there exis two bounded functions y_η and y_ρ such that :*

$$\delta\rho = y_\rho \delta\overline{T} = y_{\rho,p}\delta\overline{T}_p \mathbf{1}_{(0,1)}(\chi) + y_{\rho,f}\delta\overline{T}_f \mathbf{1}_{(1,2)}(\chi) \tag{5.23}$$

$$\delta\eta = y_\eta \delta\overline{T} = y_{\eta,p}\delta\overline{T}_p \mathbf{1}_{(0,1)}(\chi) + y_{\eta,f}\delta\overline{T}_f \mathbf{1}_{(1,2)}(\chi). \tag{5.24}$$

Then, the matrix A_2 (5.19-5.22) becomes :

$$\tilde{A}_2(\chi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{1,3} & 0 & A^*_{2,3} & A_{2,4} \\ \alpha_f \delta_{(1^-)}(\chi) & 0 & \gamma_l \left(\frac{y_\eta}{\eta} + \frac{y_\rho}{\rho} \right) & 0 \end{pmatrix} \tag{5.25}$$

with $A^*_{2,3} = A_{2,3} + \gamma_T y_\eta + \frac{\Omega_{1,e}}{\rho_e} \mathbf{1}_{(0,1)}(\chi) y_\rho$.
And the operator A is still stable as $\tilde{A}_2$ is yet a bounded perturbation of the exponentially stable differential operator with compact resolvent A_1.

Proof:
(Sketch) The open loop system (5.17) with B defined in (5.20) and $\varphi = \delta\overline{\phi}$ the open loop state variable, is given by :

$$\begin{aligned} \partial_t \varphi(\chi, t) &= A(\chi)\varphi(\chi, t) + Bu = A_1(x)\partial_\chi \varphi(\chi) + \tilde{A}_2(\chi)\varphi(\chi) + B(\chi)u(\chi, t) && (5.26) \\ y(t) &= C\varphi(\chi, t) && (5.27) \end{aligned}$$

where the output $y(t)$ is the temperature inside the barrel, and $\tilde{A}_2 = A_2 + \mathcal{C}$.
$A_1\partial_x$ generates a C_0-semigroup exponentially stable given the linearized boundary conditions (as ∂_x it is for this kind of boundary conditions) [25] . $\tilde{A}_2$ defined in (5.25) is viewed as a bounded perturbation of $A_1\partial_\chi$ [40]. The same is done with the control part $B(\chi)u(\chi, t)$ which can be also assimilated to a bounded perturbation, under regularity assumptions on $u(\chi, t)$. □

The open loop stability is preserved even if the melt density and the viscosity are supposed to be non constant and governed by laws which can be described by the evolution equations (5.23)-(5.24).

5.4.2 IMBC structure

For this problem, the control scheme based on the Internal Model Boundary Control (IMBC) [20] is again adopted considering it is a time delay system (figure (5.3)). This control strategy integrates the model of the process in real time. It allows to regulate the temperature in all the points of the barrel by taking into account the error between the linearized model and the real system (or the non linear model for the simulations).

5.4.3 Closed Loop Stability

To prove that it is still stable for the closed loop system, the same theoretical results are used : the closed loop system is viewed as a perturbation of the open loop system [40], [65]. It can be proved, that the closed loop system operator generates a C_0-semigroup under a proportional integral control

$$u(t) = \alpha_i K_i \int_0^t y(\tau) - r(\tau) d\tau + \alpha_p K_p [y(t) - r(t)]$$

with $\alpha_i,\ \alpha_p \in \mathbb{R}^+$ and $K_i,\ K_p \in \mathbb{C}^{p\times p}$, $p = dim(Y),\ y \in Y$.
Let

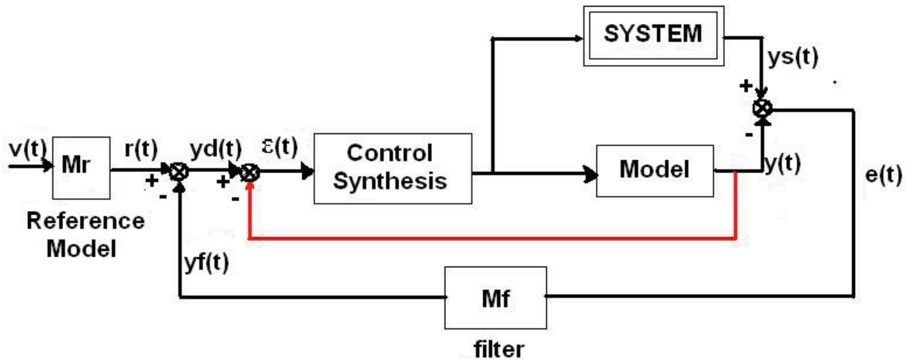

Figure 5.3 : *IMBC structure : Internal Model Boundary Control*

– $\zeta(\chi, t) = \int_0^t y(\chi,\tau) - r(\tau)d\tau = \int_0^t C\varphi(\chi,\tau) - r(\tau)d\tau$
– ξ is the closed loop state variable and $\xi = (\varphi \ \zeta)^T$

the closed loop system is, with $A = A_1\partial_x + A_2$:

$$\dot{\xi} = \begin{pmatrix} A + B\alpha_p K_p C & B\alpha_i K_i \\ C & 0 \end{pmatrix} \xi - \begin{pmatrix} B\alpha_p K_p \\ 1 \end{pmatrix} r(t)$$
$$\dot{\xi}(\chi,t) = \mathcal{A}(\chi)\xi(\chi,t) + \mathcal{B}r(t) \tag{5.28}$$

Given to [65], appropriate choices for tuning the controller parameters are, under the condition that $rg(CA^{-1}B) = p$ and that A satisfies the spectrum decomposition assumption :

– $0 \leq \alpha_p < \left(\sup_{\lambda\in\Gamma_1} a\|R(\lambda;A)\| + b\|AR(\lambda;A)\|\right)^{-1}$ where a and b are selected so that $\|BK_pCx\| \leq a\|x\| + b\|Ax\|$ for all $x \in D(A)$.
– $0 \leq \alpha_i < \min_{\lambda\in\Gamma_0}[\|B_eK_i\|^{-1}, \|R(\lambda;A_e)\|^{-1}]$
– $K_p = -[C^+B^+]^{-1}$ and $K_i = +[CA^{-1}B]^{-1}$

with $P = \frac{-1}{2\pi i}\int_\Gamma R(\lambda;A)\,d\lambda$, $PB = \begin{bmatrix} B^+ & 0 \end{bmatrix}^T$, $CP = [C^+ \ 0]$, $B_eK_i = \begin{pmatrix} 0 & B\alpha_i K_i \\ 0 & 0 \end{pmatrix}$, $A_e = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.

Those conditions are necessary and sufficient to stabilize the closed loop system and to guarantee its robustness.

Proposition 5.4.2. *Suppose that the operator of the closed loop (5.28) satisfies the conditions of rank $rg(CA^{-1}B) = p$ and the operator A satisfies the spectrum decomposition assumption. Then given the conditions on the control law coefficients cited above, the extrusion process is controllable by a proportional-integral control law, via the IMBC structure. The closed loop operator generated is stable and robust around the equilibrium state.*

Proof:

In order to ensure the stability, using the results developed by [65], two conditions have to be checked previously : $rg(CA^{-1}B) = p$ and A satisfies the spectrum decomposition assumption.

Let us note that A has a compact resolvent as it has been shown [25]. $R(0, A_1\partial_x)$ is compact, and $R(0, A_1\partial_x + A_2) = R(0, A)$ is too [19]. The spectral growth condition is satisfied as B has finite rank and the state linear system $\Sigma(A, B, -)$ is exponentially stable [14].

For the condition on the rank, here the control purpose is to regulate the temperature inside the barrel. So the dimension of Y is one. The rank of $CA^{-1}B$ is also 1 as its dimension is 1×2 and it is composed of non null operators, with the definitions of A and B above.

So the proposition of [65] can be applied to the extrusion process, and under the conditions on the control parameters, the closed loop extruded system is stable. $\square$

5.5 Simulations

The data used are the following :

Density :	$\rho_0 = 1400 kg.m^{-3}$
Viscosity :	$\eta_0 = 500 Pa/s$
Heat exchange coefficient :	$\alpha = 10410 Jm^{-2}s^{-1}K^{-1}$
Specific heat capacity :	$Cp = 3.6 * 10^3 Jkg^{-1}\ K^{-1}$
Inlet moisture :	$M_{in} = 0.25$
Inlet melt temperature :	$T_{in} = 293K$
Inlet Mass flow rate :	$F_{in} = 0.025 kg/s$
Barrel temperature :	$T_{Fe} = 330K$
Length :	$L = 2m$
Pitch length :	$\xi = 30 * 10^{-3} m$
Geometric parameter :	$B = 2.4 * 10^{-10}$
Geometric parameter :	$K_d = 0.75 * 10^{-7}$

In a first step, the melt density is supposed constant. The new equilibrium profil of the temperature $T_e(\chi)$ is improved by the introduction of a non constant viscosity (fig. (5.4)).

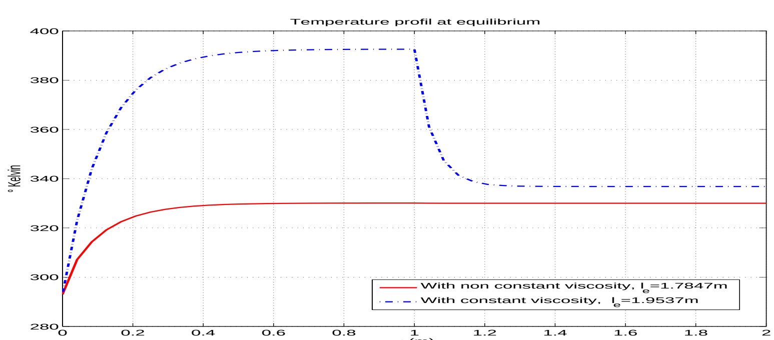

Figure 5.4 : *Temperature profil at equilibrium :* $T_e(\chi)$

The new profile obtained (red,'-') corresponds to what is physically expected.

The profile of the viscosity is given in fig. (5.5), versus the space's variable χ and versus the temperature at equilibrium T_e, in loglog scale. We can observe at left that the viscosity increases at the mixing entry and at the FFZ entry, as well. The profile is logical versus the initial conditions. In the right figure of (5.5), we can observe both phenomena of viscosity elongation and shear viscosity.

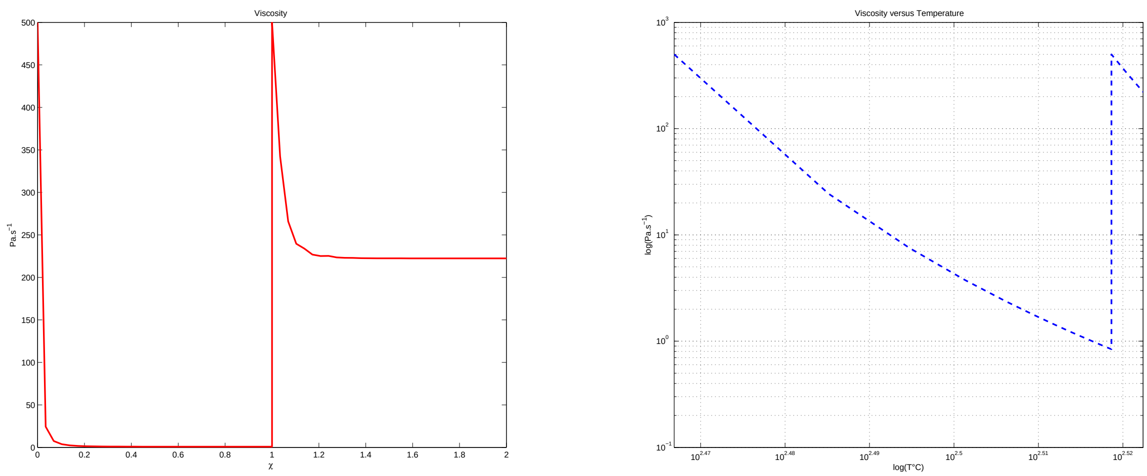

Figure 5.5 : *Viscosity profil*

The control objective is to regulate the temperature at the end of the barrel, just before the die, such that it tracks a reference signal fig. (5.6).

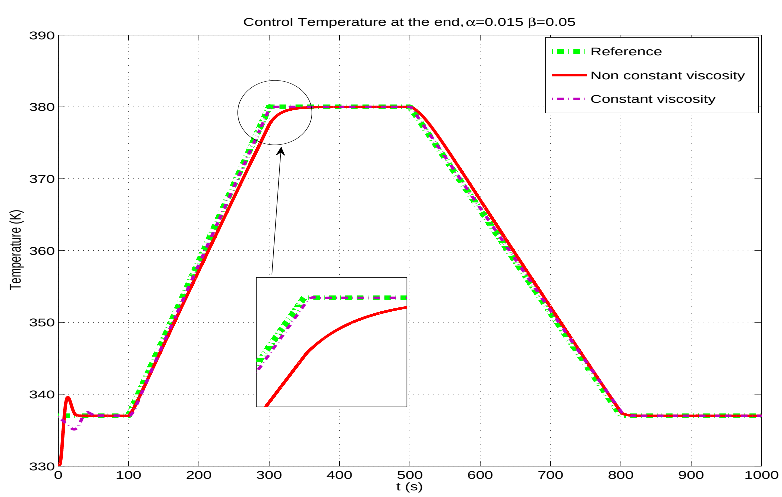

Figure 5.6 : *Control of the Temperature at $x = L$*

This figure shows that the temperature tracks the temperature desired accurately in both cases, i.e. when the viscosity is taken constant and nonconstant. The results are quite similar but the tracking is better with a constant viscosity than with a nonconstant one.

It is logical, as the added terms for the non constant viscosity act like a perturbation.

However, it is remarkable that the internal boundary l_e is improved at equilibrium by the introduction of a nonconstant viscosity, cf. fig. (5.4).

In the same order, we can observe that the evolution of this temperature inside the barrel, figure (5.7), in function of the time (each curve stands for a $\chi \in]0, 2[$), is regular and there is no temperature jumping (smooth evolution of the temperature). Here, numerical values of the gains (α_x and K_x) have been developed and the values obtained are around 5.10^{-3}. For future works, theoretical expression should be developed at least for the fixed part of the control , i.e. K_x, and should improve the stability limits. For the simulations, the operating values are at least 10 times greater.

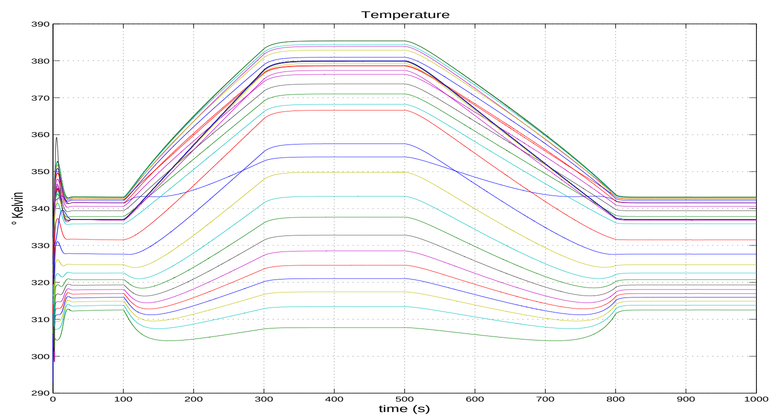

Figure 5.7 : *Evolution of the temperature inside the barrel*

A second case is developed with the tracking of the temperature, fig. (5.8), and the control implemented fig. (5.9). We can observe that the tracking is good, and the control variables have a physical meaning and a working domain, so they are realizable experimentally.

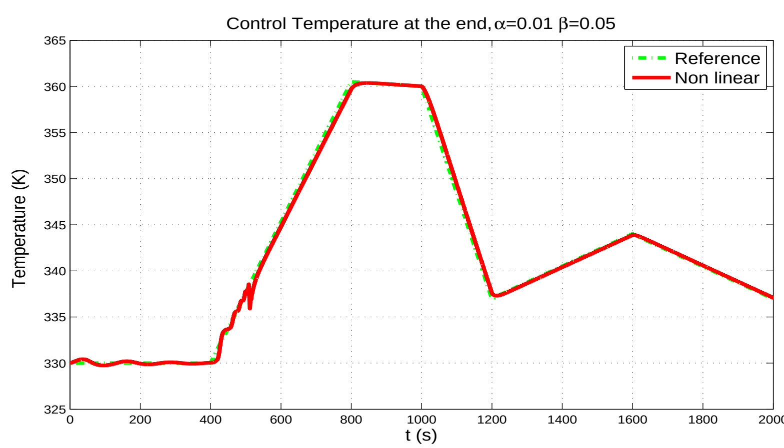

Figure 5.8 : *Evolution of the Temperature at the end of the screw*

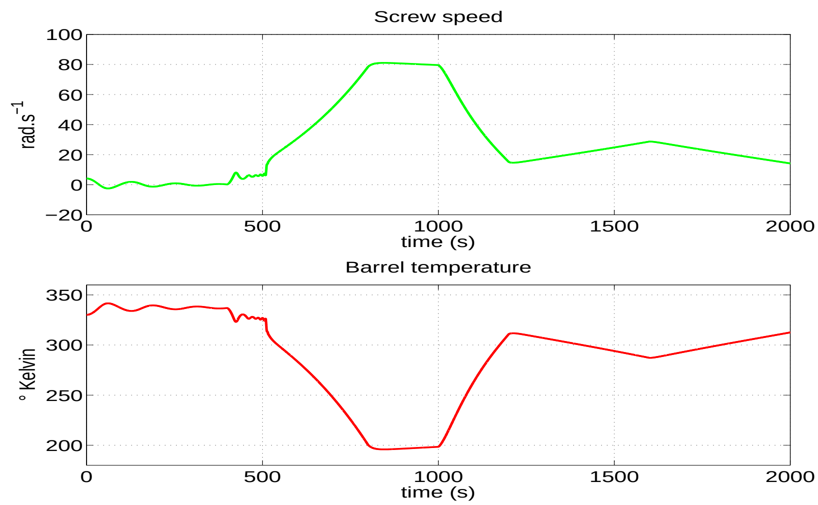

Figure 5.9 : *Evolution of the control*

In order to compare a simple feedback and the IMBC feedback, some simulations were performed emphasizing the best performance of the IMBC structure, fig. (5.10).

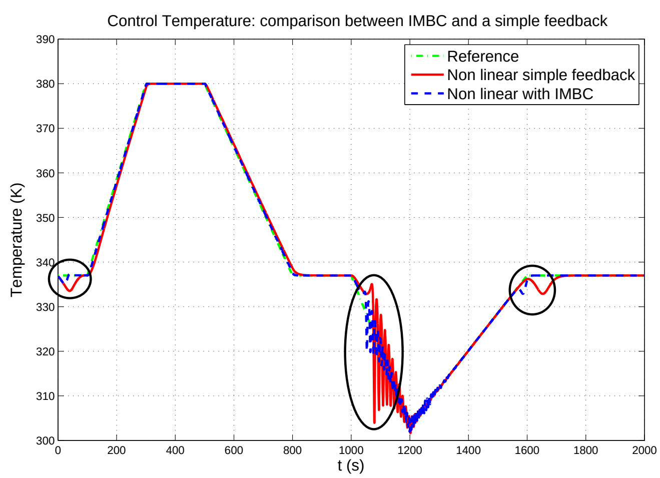

Figure 5.10 : *IMBC versus simple feedback*

5.6 Conclusion

A model of an extruder was developed, which takes into account the moving interface between the partially and the fully filled zone. The complexity of this system of coupled PDEs and ODE, comes from the mobility of the internal interface $l(t)$. Non constant laws for the melt density and the viscosity are introduced, which improve initial results for which they were constants. The linearized system in fixed coordinates is deduced, as well as the open loop properties. The closed loop stability of these equations around the equilibrium state is proved for the extrusion system, under a PI control. This control is developed using an internal model boundary control structure. The conditions for the tuning of the control parameters are given. These conditions are numerically calculated, and simulations are performed.

A important remark is that : the hypothesis that there are two phases (PFZ and FFZ), has no physical meaning for an extruder with one screw, as the extruder is always full or empty (here the inside boundary $l_e = 1.95m$). Nevertheless, this hypothesis is true for processes with two screws and the dynamics of the system thus considered are almost the same as the ones considered in the paper. So one can assume that the results developed can be transposed to an extruder with two screws.
Robustness issues and the case of extrusion process with two screws could be extensions to those works. Other controllers could be another way of investigation, example given by developing Lyapunov function via Riemann's invariants or to get a pseudo-global controller via Linear Operator Inequalities. Indeed, the proposed control is local as it has been developed around an equilibrium state. The question of interest is : what do we want to control and how ?

CHAPITRE 6

Port Hamiltonian systems

"Le secret est l'oxygène des hommes politiques et des militaires.
Il en faut toujours un petit peu pour les faire avancer."
La Trilogie du Vide, 3. Vide en évolution (2011)

Peter F. Hamilton, Romancier, nouvelliste britannique de science, 1960-...

Sommaire

6.1 Introduction

I stepped into Hamiltonians through the ANR PARADE and the PhD thesis of Redha Mulla and in parallel via the on Master research of Yves Hagedorn, whose subject under my direction was "Non-linear stability study of a unidirectional flow by Lyapunov function : a Hamiltonian approach". Initial developments were made in collaboration with Bernhard Maschke and Yann Le Gorrec, then I continued the research on my own. These collaborations work in infinite-dimensional port Hamiltonian systems (Dirac structure) and the invariance properties of some sets of variables. Initially, we started with the Casimir invariants, and very quickly we made the link with the Riemann invariants. We have demonstrated that the different structures could be linked by Cayley type transformations. I subsequently have extended these works and demonstrated that we could also link them with the Small Gain Theorem.

I also invested my efforts on the geometric approach, extension of work done on a class of nonlinear systems suitable for Irreversible Thermodynamics in the sense that they are defined on a state space with a contact form associated with the Gibbs equation. The Audrey FAVACHE thesis work focused on an alternative formulation process models where the generating function of the dynamics, Hamiltonian function, the dimension of entropy per unit time and represents a virtual entropy flows and their dynamic properties. This work is not reproduced here due to lack of place. The reader can find the related publication [33] at the end of this document.

6.2 Statement

Here we are concerned with the stabilization via boundary control of hyperbolic systems of two conservation laws. The stabilization by boundary control of irrigation channels has been intensively studied for instance in [4, 21, 22] for both linear and non linear cases. The stability of hyperbolic partial differential equations of a one-dimensional spatial domain has been intensevely studied in the literature [3, 5]. One of the most common suggested approaches is to use Riemann invariants for deriving a stabilizing boundary control [36]. In recent publications, some extensions are suggested based on the suitable choice of control Lyapunov function expressed in terms of Riemann's coordinates of the system [3, 5, 11, 12, 13].

The use of physically motivated control Lyapunov function for the derivation of stabilizing control of non-linear finite-dimensional systems has proven to be very efficient and has led to a variety of results [6, 83, 73]. Very often, when the system stems from physical modeling, one may derive dissipation inequalities related to energy balance equations and energy dissipating phenomena [81]. Using the dissipative port-Hamiltonian formulation for controlled physical systems [6, 61, 73], one may go one step further and assign in closed loop not only some dissipation inequality for some suitable control Lyapunov function, but also assign the dynamic behavior by the structure matrices of the Hamiltonian system in closed loop [57, 62]. For infinite-dimensional systems very similar techniques based on dissipation inequalities, which in terms of PDE's amounts to consider some entropy function [3, 12, 13], have been used for the stabilization of boundary control systems [14, 53]. Recent works have used a boundary port-Hamiltonian formulation of systems of conservation laws [58, 74] in order to derive stabilizing boundary control for a class of linear systems defined on one-dimensional spatial domains [47, 54, 77, 78].

The notion of dissipativity is generally linked with the Small-Gain Theorem. This link has been mentioned (e.g. [3, 4, 11, 12, 13]) several times, but not proved for Greenberg & Li's theorem [36] and the generalized version of this theorem given by [12, 13].

The aim is to describe the links stated in Fig. 6.1.

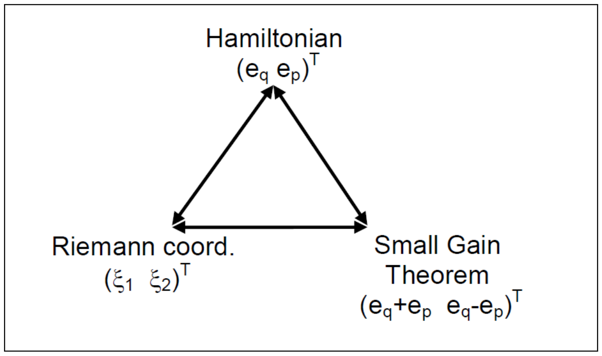

Figure 6.1 : *Relations between Hamiltonian-Riemann-Small Gain Theorem*

6.3 Port Hamiltonian formulation of a hyperbolic system of two conservation laws

6.3.1 Preliminary notions on the Riemann invariants for an hyperbolic system

Let us recall briefly the main result on the stabilization of a hyperbolic system of two conservation laws suggested by Greenberg & Li [36]. Consider a spatial domain consisting of the finite interval $[0, L] \ni x$ with $L \in I\!R_+^*$ and time domain being the real interval $[0, +\infty) \ni t$. The state space is a non-empty connected open set in $I\!R^2$, denoted by Ω. Consider the system of two conservations laws :

$$\partial_t Y + \partial_x f(Y) = 0, \tag{6.1}$$

where

- $Y = (y_1\ \ y_2)^T : [0, +\infty) \times [0, L] \to \Omega$ is the vector of the two dependent variables ;
- $f : \Omega \to I\!R^2$ is a C^1-function called *the flux vector*.

Note that the system (6.1) may also be written as :

$$\partial_t Y + F(Y)\partial_x Y = 0 \tag{6.2}$$

where F is the Jacobian of the flux vector f. The assumption that the system is hyperbolic implies that this system can be diagonalised using the Riemann invariants (see for instance [46, pages 34 - 35]). This means that there exists a change of coordinates $\xi(Y)$ whose Jacobian matrix is denoted $D(Y)$,

$$D(Y) = \frac{\partial \xi}{\partial Y}, \tag{6.3}$$

that diagonalises $F(Y)$ in Ω :

$$D(Y)F(Y) = \Lambda(Y)D(Y) \qquad Y \in \Omega.$$

In the coordinates ξ, the system (6.1) can then be rewritten in the following (diagonal) characteristic form :

$$\partial_t \xi + \Lambda(\xi)\partial_x \xi = 0 \tag{6.4}$$

with $\xi : [0, L] \times [0, +\infty) \to I\!R^2$, $(x, t) \mapsto \xi(x, t)$, and $\Lambda(\xi) = \mathrm{diag}(\lambda_1(\xi), \lambda_2(\xi))$, with $\lambda_1(\xi), \lambda_2(\xi)$ satisfying the conditions :

- the λ_i's are continuously differentiable functions on a neighborhood of the origin ;
- $\lambda_2(0) < 0 < \lambda_1(0)$.

In this part we shall consider the following result of Greenberg and Li [36].

Theorem 6.3.1. *Consider the hyperbolic system of conservation laws in Riemannian coordinates (6.4) with the following relations on the boundary variables :*

$$\xi_2(0) = \mathbf{K_1}(\xi_2(0)),\ \xi_1(L) = \mathbf{K_2}(\xi_2(L)) \tag{6.5}$$

with the functions K_1 and K_2 being C^1 and satisfying :

$$\mathbf{K_1}(0) = \mathbf{K_2}(0) = 0 \textit{ and } |\mathbf{K_1}'(0)\mathbf{K_2}'(0)| < 1. \tag{6.6}$$

Consider initial values :

$$\lim_{t \to 0^+} (\xi_1, \xi_2)(x, t) = (\xi_{1,0}, \xi_{2,0})(x),\ 0 < x < L, \tag{6.7}$$

being C^1 and satisfying the assumption that to be small in the C^1 norm and the compatibility conditions :

$$\xi_{2,0}(0) = \mathbf{K_1}(\xi_{1,0}(0)) \tag{6.8}$$

$$\xi_{1,0}(L) = \mathbf{K_2}(\xi_{2,0}(L)) \tag{6.9}$$

$$\lambda_2(\xi_{1,0}, \xi_{2,0})(0)\, \partial_x \xi_{2,0}(0) = \lambda_1(\xi_{1,0}, \xi_{2,0})(0)\, \mathbf{K_1}'(0)\, \partial_x \xi_{1,0}(0) \tag{6.10}$$

$$\lambda_1(\xi_{1,0}, \xi_{2,0})(L)\ \partial_x \xi_{1,0}(L) = \lambda_2(\xi_{1,0}, \xi_{2,0})(L)\, \mathbf{K_2}'(L) \partial_x \xi_{2,0}(L) \tag{6.11}$$

Then the initial value problem, for this system, has a unique C^1 solution. Moreover, its solution decays to zero in the C^1 norm with an exponential rate.

The generalized theorem developed by Coron, Bastin & Andréa-Novel is

Theorem 6.3.2. *[13] If $\rho_1(K'(0)) < 1$, then the equilibrium $u \equiv 0$ of the quasi-linear hyperbolic system (6.4) with the following relations on the boundary variables :*

$$\begin{pmatrix} \xi_2'(0) \\ \xi_1'(L) \end{pmatrix} = \nabla K \begin{pmatrix} \xi_1'(0) \\ \xi_2'(L) \end{pmatrix}. \tag{6.12}$$

is exponentially stable.

6.3.2 Boundary port Hamiltonian systems and Riemann coordinates

a. Hamiltonian operator expressed in the Riemann coordinates

In this section we consider a hyperbolic system of two conservation laws (6.1) which admits a Hamiltonian representation, that is such that vector of flux variables may be written as follows :

$$\partial_x\, e(Y) = \mathcal{J} \begin{pmatrix} \delta_{y_1} H \\ \delta_{y_2} H \end{pmatrix}. \tag{6.13}$$

with the canonical Hamiltonian operator

$$\mathcal{J} = \epsilon \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{6.14}$$

Hence the system is written :

$$-\partial_t Y = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} (\delta_Y H). \tag{6.15}$$

In the following we will express the Hamiltonian system in terms of the Riemann's invariants and give the expression of the Hamiltonian operator as well as of the boundary port variables. Denoting by $\tilde{H}(\xi)$ the Hamiltonian expressed in the Riemann invariants : $\tilde{H}(\xi) = H \circ Y(\xi)$ where Y denotes, with an abuse of notation, the inverse change of coordinates to the Riemann coordinates.

Let us define $D_\xi = D \circ Y(\xi)$. We obtain by multiplying both terms of (6.15) by D :

$$\begin{aligned} & -D(Y)\partial_t Y = D(Y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(\delta_Y H) \\ \Leftrightarrow \quad & -\partial_t \xi = D_\xi(\xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D^T(\xi)\delta_\xi \tilde{H}(\xi)) \\ \Leftrightarrow \quad & -\partial_t \xi = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D_\xi^T)\delta_\xi \tilde{H}(\xi) + D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\xi^T \partial_x(\delta_\xi \tilde{H}(\xi)). \end{aligned}$$

Hence in terms of the Riemann invariants the system is written :

$$-\partial_t \xi = (B\partial_x + C)\delta_\xi \tilde{H}(\xi), \tag{6.16}$$

where $B = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\xi^T$ and $C = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D_\xi^T)$. The following properties may be noted : first the matrix B is symmetric, and second, it is related with the matrix C by :

$$\partial_x B = C^T + C. \tag{6.17}$$

b. Boundary port variables

In this section we will check the formal skew-symmetry of the differential operator $(B\partial_x + C)$ and then define port boundary variables associated with it. Let us define the following bracket on smooth functions on the spatial domain $[0,\ L]$:

$$\{e_1, e_2\} = \int_0^L e_1^T (B\partial_x + C) e_2 \, dx \tag{6.18}$$

and consider the symmetric product [23] :

$$\{e_1, e_2\} + \{e_2, e_1\} = \int_0^L \partial_x(e_1^T B e_2) = \left[(e_1^T B e_2)\right]_0^L. \tag{6.19}$$

The product (6.19) corresponds to Stokes theorem applied to the equation for the differential operator $(B\partial_x + C)$. Furthermore the second member of (6.19) vanishes for all functions e_1, e_2 with compact support strictly included in the domain $[0,\ L]$ and hence for these functions the bracket is skew-symmetric.

However the bracket (6.18) is not skew-symmetric for functions which do not vanish on the boundary of the domain. In this case the time variation of the Hamiltonian becomes :

$$\frac{d\tilde{H}(\xi)}{dt} = \left[(\delta_{\xi_1} \tilde{H}^T B \, \delta_{\xi_2} \tilde{H}) \right]_0^L . \tag{6.20}$$

The definition of the boundary port variables follows strictly the construction suggested in [47]. Now the differential operator $B\partial_x + C$ completed with the definition defines the following vector space :

$$\begin{gathered} \tilde{\mathcal{D}} = \left\{ \left(\begin{pmatrix} f \\ f_\partial \end{pmatrix}, \begin{pmatrix} e \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} \; / \; f = (B\partial_x + C)e \right. \\ \left. \begin{pmatrix} e_\partial(L) \\ f_\partial(L) \\ e_\partial(0) \\ f_\partial(0) \end{pmatrix} = \mathbf{diag}\,(1,\ 1,\ -1,\ 1) \begin{pmatrix} e_1(L) \\ e_2(L) \\ e_1(0) \\ e_2(0) \end{pmatrix} \right\} \end{gathered} \tag{6.21}$$

Adapting the proofs in [47], one may prove that the vector space $\tilde{\mathcal{D}}$ is a Dirac structure with respect to the pairing defined on $((f,\ f_\partial), (e,\ e_\partial)) \in \left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \times \left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right)$:

$$\langle (f,\ f_\partial), (e,\ e_\partial) \rangle = \int_0^L e^T f \, dx + e_\partial^T f_\partial$$

which is canonical in the sense that it does not depend on the differential operator anymore [23].

6.3.3 Stabilizing boundary relations with respect to the Riemann invariants and boundary port variables

Consider a hyperbolic system of two conservation laws (6.1) which admits a Hamiltonian representation (6.15) with port variables.

Then, let us consider the relations on the boundary port variables defined by C^1 functions G_∂ and G :

$$\begin{pmatrix} e_\partial^0 \\ f_\partial^L \end{pmatrix} = G_\partial(f_\partial^0, e_\partial^L), \tag{6.22}$$

and

$$\begin{pmatrix} e_1^0 \\ e_2^L \end{pmatrix} = G(e_1^L, e_2^0) \tag{6.23}$$

The energy balance equation depends on $G_{\partial 1}, G_{\partial 2}$ (the line components of G_∂) and becomes :

$$\frac{dH}{dt} = e_\partial^L G_{\partial 2}(f_\partial^0, e_\partial^L) + f_\partial^0 G_{\partial 1}(f_\partial^0, e_\partial^L) \tag{6.24}$$

Using the implicit function theorem, the relations (6.22) on the port boundary variables, may be expressed in terms of boundary port variables (6.5) on the boundary values of the Riemann coordinates (6.12).

Notation 6.3.3.

1. *An abuse of notation is done all along the article in order to facilitate the reading. According to the cases, the notation "f'" can stand for the derivative of f or its gradient.*
2. *The following notations are used,*

$$\bar{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G$$

and in the same idea

$$\hat{K} = K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.25}$$

3. *Let us pose F the jacobian and D the matrix of changes such that*

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$DF \quad = \quad \Lambda D \Leftrightarrow D = \begin{pmatrix} a & \frac{-a(\alpha-\lambda_1)}{\gamma} \\ c & \frac{-c(\alpha-\lambda_2)}{\gamma} \end{pmatrix} \tag{6.26}$$

Remark 6.3.1. Two equivalent cases can appear, as follows :
Case n°1

$$\begin{pmatrix} e_1^L \\ e_2^0 \end{pmatrix} = G^{-1}(e_1^0, e_2^L) \tag{6.27}$$

Case n°2

$$\begin{pmatrix} e_1^0 \\ e_2^L \end{pmatrix} = G(e_1^L, e_2^0), \tag{6.28}$$

This second case is preserved, the first one is exactly the same [30].

The boundaries relations are developed by different changes of variables in Riemann coordinates/Hamiltonian coordinates and boundaries relations on Riemann formalism/Hamiltonian formalism. These transformations allow to link the boundary function K in Riemann coordinates to the boundary function G for the Hamiltonian coordinates (An idea is given in [23]).

Using the derivative of the functions K_i of the boundary conditions, (6.5) can be expressed as :

$$\nabla\hat{K} \quad = \quad [Id - \mathcal{A}]^{-1}\,[Id + \mathcal{A}] \tag{6.29}$$

with

$$\mathcal{A} = \begin{pmatrix} \frac{2a}{\lambda_1}(0) & 0 \\ 0 & \frac{c(\lambda_2-\lambda_1)}{\gamma\lambda_2}(L) \end{pmatrix} \nabla\bar{G} \begin{pmatrix} \frac{\gamma\lambda_1}{-a(\lambda_2-\lambda_1)}(0) & 0 \\ 0 & \frac{\lambda_2}{2c}(L) \end{pmatrix}$$

under the condition that $[Id - \mathcal{A}]$ is invertible irrespective of how the boundary variables are related together.
Sufficient regularity of the functions is assumed, and in order to homogenize all the results developed below the matrical case is taken (that is say ∇G and all the dependent functions are matrices).

Proposition 6.3.4. *If the components of $\nabla \bar{G}$ satisfies the following conditions*

$$tr(\mathcal{A}) = -G'_{21}(L)\frac{2\gamma}{(\lambda_2 - \lambda_1)}(L) + G'_{12}(0)\frac{(\lambda_2 - \lambda_1)}{2\gamma}(0) < 0,$$

$\mathcal{A}(0) = -Id$ *and*

$$\det \mathcal{A} > 0$$

then the spectral radius of $[Id - \mathcal{A}]^{-1}[Id + \mathcal{A}]$ *satisfies*

$$\rho([Id - \mathcal{A}]^{-1}[Id + \mathcal{A}]) < 1$$

as it is exactly a Cayley transformation of the matrix if $\mathcal{A}$ *is a closed operator [1].*
Note that $\mathcal{A}$ *is closed if* $\nabla \bar{G}$ *is invertible (not the only case). If furthermore the compatibility conditions (6.10)-(6.11) are satisfied the conditions of theorem 6.3.1 are satisfied and the system is exponentially stable (see extension of this theorem (6.3.2) [13]).*

Proof:
Let's consider a hyperbolic system of two conservation laws (6.1) which admits a Hamiltonian representation (6.15) with port variables.

Then consider the relations on the boundary port variables defined by a C^1 function G for the Hamiltonian coordinates and K for the Riemann ones.

So we get the relation (6.29)

$$\nabla \hat{K} = [Id - \mathcal{A}]^{-1}[Id + \mathcal{A}] \tag{6.30}$$

Consequently if the real part of the eigenvalues of

$$\mathcal{A} = \begin{pmatrix} \frac{2a}{\lambda_1}(0) & 0 \\ 0 & \frac{c(\lambda_2-\lambda_1)}{\gamma\lambda_2}(L) \end{pmatrix} \nabla \bar{G} \begin{pmatrix} \frac{\gamma\lambda_1}{-a(\lambda_2-\lambda_1)}(0) & 0 \\ 0 & \frac{\lambda_2}{2c}(L) \end{pmatrix}$$

are negative then $\rho(\nabla \hat{K}) < 1$. Indeed, $\nabla \hat{K}$ is a Cayley transformation of $\mathcal{A}$.

Let's calculate the eigenvalues of $\mathcal{A}$:

$$vp(\mathcal{A}) := \lambda^2 - tr(\mathcal{A})\lambda + \det(\mathcal{A}) = 0$$

Two cases can occur :

$$\begin{aligned} \Delta_{\mathcal{A}} &= tr(\mathcal{A})^2 - 4\det(\mathcal{A}) > 0 \text{ and } \lambda = tr(\mathcal{A}) \pm \sqrt{\Delta_{\mathcal{A}}} \\ &\text{or} \\ \Delta_{\mathcal{A}} &= tr(\mathcal{A})^2 - 4\det(\mathcal{A}) < 0 \text{ and } \lambda = tr(\mathcal{A}) \pm i\sqrt{\Delta_{\mathcal{A}}} \end{aligned}$$

For both cases

$$tr(\mathcal{A}) = -G'_{21}\frac{2\gamma}{(\lambda_2 - \lambda_1)}(L) + G'_{12}\frac{(\lambda_2 - \lambda_1)}{2\gamma}(0) < 0 \tag{6.31}$$

$$\det \mathcal{A} > 0 \tag{6.32}$$

if and only if $\mathcal{R}e(vp(\mathcal{A})) < 0$, the previous proposition holds true. □

6.4 Link between dissipativity /Small gain theorem

This notion of dissipativity is generally linked with the Small-Gain Theorem. This link has been mentioned (e.g. [12, 13]) but not proved for the theorems (6.3.1) and (6.3.2) and the Riemann invariants in general.

It is proposed here to link the Hamiltonian functional expressed in Riemann coordinates with the Small Gain Theorem. Indeed, we can write :

$$d_t\tilde{H}_\xi = \begin{pmatrix} \delta_\xi H^T(L) & \delta_\xi H^T(0) \end{pmatrix} \begin{pmatrix} B(L) & 0 \\ 0 & -B(0) \end{pmatrix} \begin{pmatrix} \delta_\xi H(L) \\ \delta_\xi H(0) \end{pmatrix} \tag{6.33}$$

$$= \frac{1}{2} \begin{pmatrix} e_2(L)+e_1(L) \\ e_2(0)-e_1(0) \\ e_2(L)-e_1(L) \\ e_2(0)+e_1(0) \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_2(L)+e_1(L) \\ e_2(0)-e_1(0) \\ e_2(L)-e_1(L) \\ e_2(0)+e_1(0) \end{pmatrix} \tag{6.34}$$

Let us define

$$y = \begin{pmatrix} e_2(L)+e_1(L) \\ e_2(0)-e_1(0) \end{pmatrix} \text{ and } u = \begin{pmatrix} e_2(L)-e_1(L) \\ e_2(0)+e_1(0) \end{pmatrix}$$

then

$$\begin{aligned} d_t\tilde{H}_\xi &= \frac{1}{2}\begin{pmatrix} y^T & u^T \end{pmatrix}^T \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} \\ &= \frac{1}{2}y^Ty - \frac{1}{2}u^Tu && (6.35) \\ \text{if} \quad & y = Mu && (6.36) \\ \text{then } d_t\tilde{H}_\xi &= \frac{1}{2}u^T[M^2-1]u && (6.37) \end{aligned}$$

Proposition 6.4.1. *The Small Gain Theorem allows to conclude that the system (6.37) is stable if* $\|M\| < 1$.

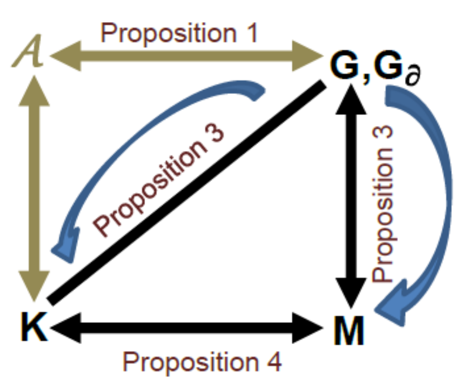

Figure 6.2 : *Relations between Hamiltonian-Riemann-Small Gain Theorem*

Remark 6.4.1. A choice has been made in (6.21) specifically to link the boundaries. Another choice can be also done by modifying the causality of the system. Let us define it.

$$\begin{pmatrix} e_\partial(L) \\ f_\partial(L) \\ e_\partial(0) \\ f_\partial(0) \end{pmatrix} = \mathbf{diag}\,(1,\ -1,\ 1,\ 1) \begin{pmatrix} e_1(L) \\ e_2(L) \\ e_1(0) \\ e_2(0) \end{pmatrix} \tag{6.38}$$

Let us consider the case of boundaries which are bonded together at $x = 0$ and $x = L$ resp., i.e. ∇K is a diagonal matrix. Under the assumptions of sufficient regularity of the functions involved and that the initial conditions are satisfied, the main results developed can be stated as :

Proposition 6.4.2. *Let us consider the operator K established in (6.12), the operator G_∂ established in (6.22) with the boundary relation (6.21) or (6.38).*
If the system is dissipative then the spectral radius of $\nabla\hat{K}$ is less than 1. In addition, if the compatibility conditions and the initial conditions are satisfied then the theorems (6.3.1) and (6.3.2) are satisfied [12, 13] as well, and the Small Gain theorem is also satisfied.

Proof:
The system is dissipative if $tr(G_\partial) < 0$ and $\det(G_\partial) > 0$. If we express G_∂ as a function of G, we obtain :

$$G_\partial = \begin{pmatrix} -G'_{12} & -G'_{11} \\ G'_{22} & G'_{21} \end{pmatrix} \tag{6.39}$$

if (6.21) is satisfied or

$$G_\partial = \begin{pmatrix} G'_{12} & G'_{11} \\ -G'_{22} & -G'_{21} \end{pmatrix} \tag{6.40}$$

if (6.38) is satisfied. For all the cases, $\det(\mathcal{A}) > 0 \Leftrightarrow \det(G_\partial) > 0$ as $sign(\det(\mathcal{A})) = sign(\det(G_\partial))$.

The second condition to satisfy, given the first proposition is to get $tr(\mathcal{A}) < 0$ which is equivalent to :

$$tr(\mathcal{A}) = G'_{21}\frac{2\gamma}{(\lambda_1 - \lambda_2)}(L) - G'_{12}\frac{(\lambda_1 - \lambda_2)}{2\gamma}(0) < 0 \tag{6.41}$$

$$\Leftrightarrow \quad G'_{21}\frac{2\gamma}{(\lambda_1 - \lambda_2)}(L) - G'_{12}\frac{2\beta}{(\lambda_1 - \lambda_2)}(0) < 0 \tag{6.42}$$

$$\Leftrightarrow \quad G'_{21}2\gamma(L) - G'_{12}2\beta(0) < 0 \tag{6.43}$$

$$\Leftrightarrow \quad G'_{21}(L)\gamma(L) < G'_{12}(0)\beta(0) \tag{6.44}$$

By definition of β and γ and the hyperbolic nature of the system, we find that $\beta\gamma > 0$, so they have the same sign.

So if β and γ are positive then, $tr(G_\partial) < 0 \Rightarrow tr(\mathcal{A}) < 0$ because the condition on the determinant, $G'_{21}(L) < 0$ and $G'_{12}(0) > 0$ and (6.44) is verified.
If β and γ are negative, it means that the way the boundaries have been linked does not respect causality and G_∂ has to be changed. In other words, the equation (6.21) becomes (6.38). Then $tr(G_\partial) < 0$ and $\det(G_\partial) > 0$ imply that $G'_{12} < 0$ and $G'_{21} > 0$ and (6.44)

is verified. The assumptions of the proposition 1 are satisfied, therefore the generalized theorem of Greenberg & Li can be applied (under sufficient regularity of the functions involved, compatibility conditions and initial conditions which are supposed to be checked).
The second part of the proposition linked with the Small Gain Theorem is obvious by the definition of ∇G (6.23) and G_∂ (6.22). It has been proved in [30]. □

Proposition 6.4.3. *Let us consider the diagonal case on the boundaries i.e. ∇K (6.12) is a diagonal matrix.*
If the operator K satisfies theorem (6.3.2) defined in [13] then the operator M defined in (6.36) satisfies the condition $\|M\| < 1$ of the Small Gain Theorem, and inversely.

Proof:
If K satisfies the generalized theorem (6.3.1) then $\rho(\nabla \hat{K}) < 1$, then $\mathcal{A}$ satisfies the conditions $tr(\mathcal{A}) < 0$ and $\det(\mathcal{A}) > 0$, then $tr(G_\partial) < 0$ and $\det(G_\partial) > 0$.

As for the expression (6.29), the relation between operators M and G can be expressed as follows :

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla G - I_2 \right] \left[I_2 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla G \right]^{-1} \tag{6.45}$$

and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M$ is a Cayley transformation of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla G$.
All the more $tr\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla G \right) = tr(G_\partial)$ and the determinants are equals too.

It is clear here that $\rho(M) < 1$ and the Small Gain Theorem can be applied.
The only point that can be obscure at the first view is the passage from $tr(\mathcal{A}) < 0$ and $\det(\mathcal{A}) > 0$ to $tr(G_\partial) < 0$ and $\det(G_\partial) > 0$. But for the diagonal case, it is quite obvious. Indeed the assumptions on the determinant are clear as they have the same sign. It means that $-G'_{21}(L)G'_{12}(0) > 0$ and so $G'_{21}(L)$ and $G'_{12}(0)$ have opposite sign and they satisfy $G'_{21}(L)\gamma(L) < G'_{12}(0)\beta(0)$ if (6.21) and β and γ are positive. So $G'_{21}(L) < G'_{12}(0)$ and $tr(G_\partial) < 0$ (respectively for the boundary conditions (6.38)).
So under assumptions on the initial conditions and existence, the theorem is satisfied. □

The last proof shows the main problem to go from the stability condition developed for the Riemann formalism and the Small Gain Theorem : the passage from the hypotheses on $\mathcal{A}$ and G_∂ is not obvious if the diagonal case is not treated.

6.5 Shallow water equations

6.5.1 Application to the shallow water equations

We consider the special case of a reach of an open channel delimited by two underflow gates as represented in Figure (6.3).

We assume that the channel is horizontal, prismatic with a constant rectangular section and a width, and that the **friction effects are neglected**.

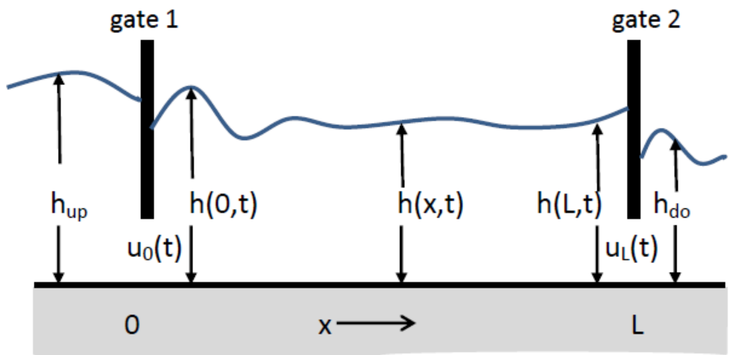

Figure 6.3 : *A reach of an open channel delimited by two adjustable underflow gates*

The flow is the canal may be described by the so-called shallow water equations or de Saint-Venant equations which constitute a system of two conservation laws, actually the mass balance and the momentum balance equations :

$$\partial_t h + \partial_x(hv) = 0, \tag{6.46}$$

$$\partial_t v + \partial_x(\frac{1}{2}v^2 + gh) = 0, \tag{6.47}$$

when v denotes the velocity of the water flow and the water level h. Each underflow gate imposes a boundary condition of the form :

$$\begin{aligned} Wh(0,t)v(0,t) &= U_0(t)\Psi_1(h(0,t)), & (6.48)\\ Wh(L,t)v(L,t) &= U_L(t)\Psi_2(h(L,t)), & (6.49) \end{aligned}$$

where W is the channel width and $\Psi_1(h(x,t)) = \alpha_0\sqrt{2g(h_{up} - h(x,t))}$, and $\Psi_2(h(x,t)) = \alpha_L\sqrt{2g(h(x,t) - h_{do})}$, $h_{do} < h(L)$ and $h(0) < h_{up}$ where h_{up} is the water level before the upstream gate, h_{do} is the water level after the downstream gate. α_0 and α_L are the product of the gate (or overflow) width and water-flow coefficient of the corresponding gate. U_0 and U_L are the control functions.

a. Hamiltonian formulation of the shallow water equations

The Hamiltonian formulation is briefly presented, it is described in detail in the general case (with slope and friction) in [37], see also [64]. The energy of the water flow in the channel is defined in terms of two state variables :

- the momentum $p(x,t) = \rho v(x,t)$ where ρ the mass density of the water (constant as the water is assumed to be incompressible),
- the section area of the water $q(x,t) = W\,h(x,t)$.

The state vector is defined by $Y = \begin{pmatrix} q & p \end{pmatrix}^T$. The total energy of the systems (sum of kinetic and potential energies) is given by :

$$H(Y) = \frac{1}{2}\int_0^L \frac{\rho g}{W}q^2 + \frac{1}{\rho}qp^2 dx, \tag{6.50}$$

where g denotes the gravitational constant. The variational derivatives of the Hamiltonian functional define the two co-energy variables; e_1 the volumic flow and e_2 the hydrodynamic pressure given as follows :

$$e_2 = \delta_q H(Y) = \frac{p^2}{2\rho} + \frac{\rho g}{W} q \left(= \frac{1}{2}\rho v^2 + \rho g h\right), \tag{6.51}$$

$$e_1 = \delta_p H(Y) = \frac{qp}{\rho} \,(= W h v)\,. \tag{6.52}$$

It may be shown [37, 64] that the shallow water equations may be expressed as a system of two conservation laws (6.1) in canonical interaction and admits a Hamiltonian formulation (6.15) by writing the flux vector :

$$f(Y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_q H \\ \delta_p H \end{pmatrix} = \begin{pmatrix} \frac{pq}{\rho} \\ \frac{\rho g}{W} q + \frac{p^2}{2\rho} \end{pmatrix}. \tag{6.53}$$

Finally the Hamiltonian system may be completed with port boundary variables according to the section b. :

$$\begin{pmatrix} e_\partial^0(t) \\ e_\partial^L(t) \\ f_\partial^0(t) \\ f_\partial^L(t) \end{pmatrix} = \begin{pmatrix} \frac{g\rho}{W} q(0) + \frac{p^2(0)}{2\rho} \\ -\frac{g\rho}{W} q(L) - \frac{p^2(L)}{2\rho} \\ \frac{p(0)q(0)}{\rho} \\ \frac{p(L)q(L)}{\rho} \end{pmatrix}$$

The energy balance is then expressed by :

$$\frac{dH}{dt} = e_\partial^T f_\partial = -\left[\frac{\rho g}{W} q(L) + \frac{1}{2\rho} p^2(L)\right] \left[\frac{q(L)p(L)}{\rho}\right] + \left[\frac{\rho g}{W} q(0) + \frac{1}{2\rho} p^2(0)\right] \left[\frac{q(0)p(0)}{\rho}\right]$$

b. Expression in Riemann coordinates

Using the Jacobian of the vector of flux variables

$$\nabla f(Y) = F(Y) = \begin{pmatrix} \frac{p}{\rho} & \frac{q}{\rho} \\ \frac{g\rho}{W} & \frac{p}{\rho} \end{pmatrix}$$

one obtains the following Riemann invariants :

$$\xi = \xi(Y) = \begin{pmatrix} \frac{p}{\rho} + 2\sqrt{\frac{gq}{W}} \\ \frac{p}{\rho} - 2\sqrt{\frac{gq}{W}} \end{pmatrix} \tag{6.54}$$

The Jacobian of this change of coordinates is :

$$D(Y) = \begin{pmatrix} \sqrt{\frac{g}{Wq}} & \frac{1}{\rho} \\ -\sqrt{\frac{g}{Wq}} & \frac{1}{\rho} \end{pmatrix} \tag{6.55}$$

and the eigenvalues of the system are :

$$\Lambda(\xi) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{p}{q} + \sqrt{\frac{gq}{W}} & 0 \\ 0 & \frac{p}{q} - \sqrt{\frac{gq}{W}} \end{pmatrix}$$

According to the Section a., the Hamiltonian system is expressed in the Riemann invariants as (6.16) with :

$$B = D\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}D^T = \begin{pmatrix}\frac{2}{\rho}\sqrt{\frac{g}{Wq}} & 0\\ 0 & -\frac{2}{\rho}\sqrt{\frac{g}{Wq}}\end{pmatrix}$$

$$C = \begin{pmatrix}-\frac{1}{2\rho}\sqrt{\frac{g}{Wq}}\frac{q'}{q} & \frac{1}{2\rho}\sqrt{\frac{g}{Wq}}\frac{q'}{q}\\ -\frac{1}{2\rho}\sqrt{\frac{g}{Wq}}\frac{q'}{q} & \frac{1}{2\rho}\sqrt{\frac{g}{Wq}}\frac{q'}{q}\end{pmatrix}.$$

6.5.2 Shallow water equations, dissipativity and stabilization

We took the problem of de Saint-Venant and proposed to apply the result proposition (6.3.4) to the co-variables of energy.

Let us write the co-variables of energy in Riemann coordinates, with :

$$p = \frac{\xi_1+\xi_2}{2}\rho, \quad q = \left(\frac{\xi_1-\xi_2}{4}\right)\frac{W}{g}$$

then

$$d_t\tilde{H}(\xi) = -e_2(L)e_1(L) + e_2(0)e_1(0) \tag{6.56}$$

$$d_t\tilde{H}(\xi) = e_1(L)G'_{21}(L)e_1(L) - e_2(0)G'_{12}(0)e_2(0) \tag{6.57}$$

For the shallow water equations, one gets [23], with

$$L_e = \begin{pmatrix}0 & \frac{2}{\rho\lambda_2(L)}\\ \frac{2}{\lambda_1(0)}\sqrt{\frac{g}{Wq(0)}} & 0\end{pmatrix}$$

and

$$R_i = \begin{pmatrix}0 & \frac{\rho\lambda_1(0)}{2}\\ \frac{-\lambda_2(L)}{2}\sqrt{\frac{Wq(L)}{g}} & 0\end{pmatrix}$$

$$\nabla K = \left[Id - L_e\nabla\bar{G}R_i\right]^{-1}\left[Id + L_e\nabla\bar{G}R_i\right]$$

with

$$L_e\nabla\bar{G}R_i = \begin{pmatrix}\frac{-1}{\rho}\sqrt{\frac{Wq}{g}}G'_{21}(L) & 0\\ 0 & \frac{1}{\rho}\sqrt{\frac{Wq}{g}}G'_{12}(0)\end{pmatrix}$$

which is exactly the expression of $\mathcal{A}$.

The proposition can be applied and the stability is ensure for all boundary conditions on the skew diagonal form deduced from (6.27) :

$$d_tH = \begin{pmatrix} f_\partial(L) & e_\partial(0)\end{pmatrix} G_\partial(L)\begin{pmatrix} f_\partial(L)\\ e_\partial(0)\end{pmatrix} \tag{6.58}$$

A condition to get $d_tH < 0$ that is say the dissipativity is $G'_{21}(L) < 0,\ G'_{12}(0) > 0$. Then the proposition is satisfied taking the causality as in (6.38) (see the energy expressions (6.56) and (6.58)) and the spectral radius of ∇K is less than 1 [23]. All the more the Small gain theorem is also satisfied. And reciprocally, that is say, if $\rho(\nabla K) < 1$ then one gets the dissipativity [23] and the Small gain theorem can be applied.

6.6 Conclusion

In the first part, we have recalled the expression of the port Hamiltonian systems of two conservation laws and derived its expression in terms of Riemann's invariant. The Hamiltonian operator is derived in these coordinates. The expression of the boundary port variables of the Hamiltonian formulation has also been derived. In the second part, the stability conditions on the boundary values of the Riemann invariants are expressed. As a consequence we have given an interpretation of the stabilizing boundary relations in terms of the dissipation of the Hamiltonian function on the boundary of the system. In the third part, the Small-Gain theorem is applied to the Hamiltonian functional. The link between the dissipativity given in Riemann's coordinates and the condition issued from the Small-Gain theorem is established reciprocally for the diagonal case. All of those results are applied to the shallow water equations.

The future works include defining the conditions in order to prove the inverse relation going from the Small-Gain theorem, to the dissipativity relations for a global operator not necessarily diagonal. Another question could be to see if this approach really allow to get others type of controllers.

CHAPITRE 7

Projets de recherche/ Research projects

"La technique est moins importante que les hommes ou que la société, l'important, c'est le projet humain qui est derrière."

Extrait de Internet et après.
Dominique Wolton, 1947-..., docteur en sociologie.

Sommaire

Ce chapitre décrit mon projet de recherche à moyen et long termes. Il se situe dans la continuité de mon activité passée et s'articule autour des différentes thématiques de recherche décrites dans mon mémoire à savoir la commande et l'étude de stabilité des systèmes à paramètres distribués.

Plusieurs axes ont été initiés depuis l'an 2000, mais depuis l'an passé et mon séjour en délégation au GIPSA-lab, de nouvelles voies sont en cours de développement.

7.1 Les eaux peu profondes à surface libre

Le contrôle des canaux d'irrigation représente un grand intérêt d'un point de vue écologique, économique mais aussi d'un point de vue théorique. L'eau devenant un enjeu de plus en plus stratégique et vital pour de nombreuses populations dans le monde, il est à mon sens important que l'on fasse un effort afin de préserver et de gérer au mieux cette ressource rare et précieuse.

7.1.1 Axe Multi-Modèles en dimension infinie

Depuis 2007, nous développons cet axe de recherche avec Mickael Rodrigues (Maître de Conférences au LAGEP) afin de concevoir de nouveaux régulateurs en dimension finie et en dimension infinie pour les écoulements à surface libre.
Nous avons, de par nos travaux dans ce domaine du contrôle des eaux à surface libre, développé une stratégie Multi-Modèles qui a montré de nets progrès sur la commande du niveau d'eau dans un canal tant pratique que théorique. En effet, les simulations sur les canaux de Valence et de Gignac ont été confortées par les résultats expérimentaux de nos nouveaux régulateurs. Nous avons notamment montré des résultats avec des fonctions de pondération constantes par morceaux, mais aussi avec de nouvelles fonctions de pondérations continues.

Je pense aussi aux travaux menés par le Professeur V. Puig, sur le contrôle des systèmes de grande dimension et notamment sur la commande des canaux ouverts et des eaux usées de Barcelona (projet ANR déposé en 2013-2014). Il sera utile de continuer à discuter du potentiel de nos recherches respectives afin de voir les interactions possibles et les améliorations substantielles que nous pourrions en tirer. Les domaines d'application pourront donc être les canaux d'irrigation et les eaux usées. De plus, l'étude de la stabilité et de la commande des systèmes EDP reste un problème ouvert. L'utilisation des LMI dans la synthèse de ces régulateurs pourra être utile en ayant connaissance de ce que l'on peut déjà faire en dimension finie afin d'essayer de le transcrire au niveau de la dimension infinie.

La généralisation de notre approche Multi-Modèles pour tous les systèmes EDP est un verrou scientifique qu'il est important d'explorer tant de nombreux systèmes comme les CSTR (Continuous Stirred Tank Reactor), extrudeuse, le trafic routier...sont modélisés sous cette forme. L'impact de notre méthode, au-delà de la simple application des canaux, permettrait de proposer une nouvelle façon de contrôler les systèmes décrits par des EDP hyperboliques au travers de multiples points de fonctionnement.

7.1.2 Généralisation des équations dites de Saint-Venant

a. Approche Hamiltonienne

Cet axe développé initialement avec B. Maschke présente un intérêt tout particulier de par son interprétation physique. En effet, les variables étant établies sur l'énergie et la puissance, la généralisation à d'autres types de procédés décrits par des EDP hyperboliques devraient se faire aisément.

La passerelle développée avec le théorème du Petit gain et avec les invariants de Riemann devrait permettre l'implémentation de nouveaux contrôleurs pour ce type de systèmes en dimension infinie, ou peut-être de montrer la limite pour la synthèse de commandes de tels procédés. Un développement possible serait de définir une commande améliorant celle développée via les invariants de Riemann.

Un autre axe serait d'appliquer cette approche à des systèmes non linéaires et/ou non homogènes. Les équations de Saint-Venant se prêtent bien à l'exercice, car leur complexité évolue graduellement en fonction des conditions que l'on impose.

b. Approche Riemanienne

Nous avons avec mon collègue Christophe Prieur commencé à mettre en place l'approche riemannienne pour un système d'eau à surface libre incluant une troisième équation sur la dynamique de la turpitude dans les canaux due aux algues. En effet, dans la gestion des voies d'eaux, nous devons prendre en compte les contraintes naturelles comme l'accroissement de la faune et de la flore. Il est difficile de "nettoyer directement" la flore des canaux qui est source de grandes perturbations sans affecter directement toute la faune.

Une idée est de faire des "lâchés d'eaux". Comprendre, stabiliser et gérer ces lâchés est également, avec la gestion de l'eau, un enjeu important. Nous avons à ce jour définis des conditions de stabilité, que nous souhaitons tester en simulation pour conforter la pertinence de notre approche.

c. Approche numérique

Comme précédemment, dans le cadre de ma délégation au GIPSA-lab, nous avons initié des travaux sur les performances de schémas numériques avec les professeurs Didier Georges et Gildas Besancon. Initialement nous souhaitions discuter de la gestion (toujours de l'eau) non pas de canaux mais de deltas en hydrologie (avec une extension sur le traitement de données possible).

Mes collègues ont dans le passé développé des schémas numériques pour les équations dites de Saint-Venant basés sur la méthode de Galerkin applicable pour un modèle linéarisé et sur la méthode de Boltzmann sur réseaux (LBM) pour les systèmes non-linéaires. Pour ma part, j'ai développé une méthode basée sur l'approche Chang-Cooper initialement construite pour les systèmes granulaires. Nous souhaitons et avons commencé à comparer ces méthodes sur des données réelles en boucle ouverte.

7.2 Développement de méthodologies d'identification et de commande pour des classes de systèmes de dimension infinie et finie

Ce dernier axe part également d'une collaboration qui a été le point de départ de ma demande de départ en délégation, collaboration initiée avec le professeur Ioan Landau, qui englobe aujourd'hui Luc Dugard, Delphine Bresch-Pietri et actuellement en discussion avec le professeur Miroslav Krstic.

Un grand effort de développement des techniques de modélisation, d'identification et de commande a été fait dans le domaine des systèmes de dimension finie (ou approximés comme tels). Ces techniques ont souvent donné lieu à des tests sur des plateformes expérimentales pour leur validation. Des " benchmarks " expérimentaux ont permis des confrontations objectives des différentes approches et un raffinement des techniques.

On peut affirmer que pour beaucoup de classes de systèmes de dimension finie, des techniques d'identification et de commande ont atteint un certain degré de maturité et ont été utilisées dans des nombreuses applications.

Les développements technologiques (en particulier pour les systèmes mécaniques) ont entrainé une augmentation significative de la taille des modèles de commande. Cette augmentation est dans de nombreuses situations le reflet du passage de la commande d'un vrai système de dimension finie vers un système de dimension infinie. Dans la plupart des cas, cette démarche de commande a un caractère "ad hoc" et il n'est pas certain qu'elle ne rencontrera pas des difficultés dans le futur face à des systèmes qui deviennent de plus en plus de "dimension infinie".

Trois exemples : modélisation et commande des éoliennes, commande des structures souples, contrôle actif de vibrations dans les grandes structures mécaniques.

La stratégie actuelle de commande de régulateurs part du constat qu'en général, quel que soit le système (dimension finie ou dimension infinie), il est très difficile d'obtenir un modèle de commande à partir du modèle dynamique de "connaissance" pour 2 raisons :

1. précision insuffisante sur les paramètres physiques
2. quasi-impossibilité d'obtenir des modèles fiables et numériquement robustes par les procédures classiques de simplification des modèles mathématiques.

C'est donc l'identification directe à partir des données sous un protocole d'expérimentation qui permet d'obtenir un "modèle de commande" fiable pour le calcul des régulateurs.

Si l'estimation de l'ordre de ces modèles pour des systèmes de dimension finie ne pose pas trop de problèmes, la situation est différente si on veut obtenir un tel modèle pour des systèmes de dimension infinie. Sans aucun doute, une étude des propriétés fréquentielles d'un modèle physique est nécessaire pour le choix de la fréquence d'échantillonnage et du protocole d'acquisition de données (avec dimensionnement de la séquence d'excitation). Actuellement, à notre connaissance il n'y a pas d'étude systématique sur ces aspects.

Le calcul des régulateurs à partir du modèle identifié et la prise en compte des incertitudes de modélisation (commande robuste) conduit à des régulateurs dont la taille est en général de l'ordre de 2 fois la taille du modèle (l'adjonction des contraintes de robustesse conduit à l'augmentation de la taille du régulateur). Ceci conduit in fine à des régulateurs d'ordre très élevé et à des procédures de réduction de régulateurs préservant les propriétés essentielles de la boucle fermée qui doivent être utilisées.

Toutefois, si on dispose de modèles fiables de type EDP, il est possible de concevoir directement des régulateurs de taille raisonnable. A ce jour, plusieurs régulateurs ont été développés dans le cas d'écoulement fluide et/ou visqueux en dimension infinie dans leur conception et leurs applications sur benchmark, et transposés en dimension finie dans le cadre des simulations. L'approche semi-groupe, l'approche par les coordonnées de Riemann, l'approche hamiltonienne, l'approche LOI ont permis à ce jour de synthétiser des régulateurs simples mais efficaces sur des systèmes d'EDP hyperboliques et paraboliques pour certaines approches. Dans le cadre des invariants de Riemann, il a été démontré que les régulateurs développés en dimension infinie se ramenaient à des commandes classiques en dimension finie.

Afin de définir un cadre de travail, l'exemple des forages pétroliers a été choisi et un étudiant de M2R vient de débuter son stage sur ce thème. J'ai proposé un sujet de thèse au LAGEP en collaboration avec le professeur en Génie des Procédés Mélaz Tayakout.

This chapter describes my research project at more or less long term. This is a continuation of my past activities and it focuses on different research themes described in this report, namely control and stability study of distributed parameter systems.

Several areas have been initiated since 2000, but since last year (I was in delegation with GIPSA-lab's lab), new topics are being developed.

7.3 Shallow water equations

Control of irrigation channels is of great interest ecologically, economically but also from a theoretical point of view. The water becomes an increasingly strategic and vital issue for many people in the world, it is my strong sense that we have to make an effort to preserve and to better manage this scarce and valuable resource.

7.3.1 Multi-Model axis in infinite dimension

Since 2007, we have developed this research with Mickael Rodrigues (professor assistant at LAGEP-lab) to design new regulators finite dimensional and infinite-dimensional for shallow water equations.
We have developed, through our work in this area, a Multi-Models strategy that showed significant progress in the control of the water level in a channel both from a practical and theoretical point of view. Indeed, simulations on the canals of Valencia and Gignac were supported by the experimental results of our new regulators. In particular, we showed results with constant weighting piecewise's functions, but with new continuous weight functions .

I also think about the work done by Professor V. Puig, on the control of large systems and particularly on the control of open channels and wastewater Barcelona (ANR project submitted in 2013-2014). It will be useful to continue to discuss the potential of our respective research to see the possible interactions and substantial improvements that we could draw. The application areas will be irrigation channels and wastewater. In addition, the study of the stability and control of PDE systems remains an open problem. The use of LMI in the synthesis of these regulators may be helpful with knowledge of what we can already do in finite dimension to attempt to transcribe at the infinite dimension.

The generalization of our Multi-Model approach for all PDE systems is a scientific lock which is important to explore as many systems as CSTR, extruder, traffic ... are modeled in this form. The impact of our method scope beyond the simple application of channels, would provide a new way to control systems described by hyperbolic PDE across multiple operating points.

7.3.2 Generalization of the Shallow water equations

a. Hamiltonian approach

This axis initially developed with B. Maschke is of particular interest because of its physical interpretation. Indeed, the variables are established on energy and power, the generalization to other types of processes described by hyperbolic PDE should be done easily.

The gateway developed with the Small Gain Theorem and Riemann invariants should allow the implementation of new controllers for such systems in infinite dimension, or perhaps to show the limit for the synthesis of such processes controls. A possible development would be to define a command improving one developed via Riemann invariants.

Another focus would be to apply this approach to nonlinear systems and/or nonhomogeneous. The example of the shallow water equations lend themselves to the study because of their complexity gradually evolves depending on the conditions that are imposed.

b. Riemanian approach

We have with my colleague Christophe Prieur started to set up the Riemannian approach to the shallow water system including a third equation of the dynamics of turpitude in the channels due to algae. Indeed, in the management of water channels, we must take into account the natural constraints such as increasing fauna and flora. It is difficult to "clean directly" flora channels which is a source of great disturbances without directly affecting all wildlife.

One idea is to make "releases" of waters. Understand, stabilize and manage these releases is also, with the management of water, an important issue. We have so far defined conditions of stability, we want to simulate that process to reinforce the relevance of our approach.

c. Numerical approach

As before, in the context of my delegation at GIPSA-lab, we initiated work on numerical schemes performances with professors Didier Georges and Gildas Besancon. Initially we wanted to discuss management (still water) not of channels, but of Hydrology deltas (with an extension on processing data).

My colleagues have in the past developed numerical schemes for shallow water equations based on Galerkin method applicable to a linearized model and a Boltzmann method networks (LBM) for non-linear systems. For my part, I have developed a method based on Chang-Cooper approach originally built for granular systems. We wish and started to compare these methods on real data in open loop.

7.4 Developments of identification and control methodologies for class of systems in infinite and finite dimension

This axis is issued from a collaboration that was the starting point of my application to go in delegation, collaboration initiated with Professor Ioan Landau, which includes today Luc Dugard, Delphine Bresch-Pietri and is currently in discussions with the Professor Miroslav Krstic.

The development of electro-mechanical systems in the recent years has shown a tremendous change of the dynamic of the associated mechanical structures. The reduction of the weight of the mechanical structures has led to the increased of the number of low damped vibrations modes. Appropriate finite dimensional discrete time control methodology has been developed for such situations. In addition of the weight reduction, the dimension of the mechanical structures has increased drastically (ex. : wind turbines, oil/gas drilling). The nature of the dynamic model representation has fundamentally changed, since they are becoming infinite dimensional and the dynamic models are represented by PDE. While using a "macroscopic" approach, previous developed digital control approach can be still used in some cases, it is clear that in general as dimension increases there is a limit for their use.

This calls for the development of a new control methodology (including system identification, control design, controller implementation) taking into account the PDE representation of the dynamic model associated with the mechanical structures. While previous digital control approaches grounded on, a macroscopic modeling can still be used in some cases, it is clear that in general, as dimension increases, there is a limit for their use. Since the objective is to develop and evaluate a new control methodology it is necessary to validate it on a representative experimental test bench.

The objective is to develop and evaluate an "infinite dimensional approach" for the control of large electro-mechanical structures and to compare it with a "macroscopic" finite dimensional approach.

To define a framework, the example of oil drilling has been selected and a student of M2R just started his training on this topic. I proposed a thesis in LAGEP's lab in collaboration with Professor in Process Engineering Mélaz Tayakout.

Conclusion générale

"Les hommes passent leur vie à la recherche d'eux même.
On n'arrive jamais à une conclusion définitive en ce domaine."

Rosa Candida (2010).
Audur Ava Olafsdottir, 1958-..., Ecrivain islandaise.

Ce document présente l'ensemble de mon activité d'enseignant chercheur. On retrouve en particulier dans la première partie l'ensemble des tâches d'enseignement, les responsabilités administratives et scientifiques, l'encadrement doctoral, les projets de recherche, les collaborations internationales. Dans la deuxième partie sont regroupées en trois chapitres mes activités de recherche sur la stabilité et la commande de systèmes à paramètres distribués via différentes techniques plus ou moins classiques et/ou physiques. Le dernier chapitre de mon mémoire présente mes projets de recherche.

Du fait de ma formation, des rencontres, de mes travaux depuis le DEA, j'ai développé mes compétences scientifiques autour de la stabilité et de la commande des systèmes décrits par des EDP hyperboliques. C'est donc naturellement que l'ensemble de mon activité passée et de mes projets de recherche convergent vers cette thématique.

La précision exigée sur l'étude des différents procédés fait sans aucun doute partie des défis scientifiques actuels et à venir. L'ensemble des actions de recherches de mon projet s'oriente donc en ce sens.

Certains de mes projets ont la chance de se faire en partenariat avec d'autres laboratoires français et internationaux. L'habilitation à diriger des recherches me permettra d'acquérir une autonomie supplémentaire dans la gestion et l'animation de ces projets et d'en assumer toutes les responsabilités d'encadrement et d'animation qui en découlent.

Valérie DSM

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"*Quand il s'agit d'histoire ancienne, on ne peut pas faire d'histoire parce qu'on manque de références. Quand il s'agit d'histoire moderne, on ne peut pas faire d'histoire, parce qu'on regorge de références.*"

de Charles Péguy, Extrait de Clio

[83] R. Sepulchre, M. Janković and P. Kokotović, *Contructive Nonlinear Control,* "Communications and Control Engineering", Springer, (1997). ISBN 3-540-76127-6.

Selection d'articles

"Elle a quelque chose de fou, cette exigence qui oblige chacun à amasser pour soi seul les articles de sa pensée et de sa foi ; c'est comme si chacun devait construire tout seul la ville dans laquelle il vit.
Le Territoire de l'homme - Réflexions 1942-1972 (1974)
Elias Canetti, Ecrivain britannique (1905-1994)
d'origine bulgare et d'expression allemande, prix Nobel de littérature 1981.

Sommaire

Design of a PI Control using Operator Theory for Infinite Dimensional Hyperbolic Systems

Valérie Dos Santos Martins, Yongxin Wu, Mickael Rodrigues

Abstract

This paper considers the control design of a nonlinear distributed parameter system in infinite dimension, described by the hyperbolic Partial Differential Equations (PDEs) of de Saint-Venant. The nonlinear system dynamic is formulated by a Multi-Models approach over a wide operating range, where each local model is defined around a set of operating regimes. A new Proportional Integral (PI) feedback is designed and performed through Bilinear Operator Inequality (BOI) and Linear Operator Inequality (LOI) techniques for infinite dimensional systems. The new results have been simulated and also compared to previous results in finite and infinite dimension, in order to illustrate the new theoretical contribution.

Index Terms

PDEs, de Saint-Venant Equations, Multi-Models, Semigroup Theory, IMBC.

I. Introduction

Regulation of irrigation channels has received an increasing interest over the last three decades. Water losses in open channels are very large due to inefficient management and control. To avoid overflows and satisfy the water request, the level of instrumentation (e.g., operating motor-driven gates, water level measurements) and automation in open channel networks increase [23]. In order to deliver water, it is important to ensure that the water level and the flow rate in open channels remain at certain values. The difficulty of this regulation problem is that only the gates positions can meet performance specifications. Such problems can be solved by designing boundary control laws in order to satisfy the control objectives: to maintain water level or flow rate at given values.

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The open surface channels couple transport phenomena and delay phenomena, so they have a complex nonlinear dynamics. In this case, the distributed parameter systems have a dynamic represented by hyperbolic Partial Differential Equations (PDE): the equations of de Saint-Venant, which depend on time and space [22], [26], [34]. Some studies take into account the uncertainties and apply a robust control approach [21], [20]. Studying directly the nonlinear dynamics is also possible as in [9], [13], [20], [35]. The Riemann approach has also been used to prove stability results for systems of two conservation laws [17], and for systems of larger dimensions in [19]. Recently, it has been also coupled with LMI [**?**]. The Lyapunov techniques have been used in [4], [8], [9].

In practice, industrial processes such as mining, chemical or water treatment processes are characterized by complex systems which operate in multiple operating regimes. Multi-Models methods split the operating range of a system into separate regions where local models are affected to each region [25] for control and Fault Diagnosis purposes [2], [15], [27]. Each local model is defined as a Linear Time Invariant (LTI) model for each operating point. The Multi-Models philosophy is based on weighting functions which ensure the transition between the different local models. Some authors speak about gain scheduling strategy for example in [18] or Linear Parameter Varying (LPV) controllers [29].

The use of Multi-Models representation for the study of the stability of a system described by nonlinear PDE has been examined in [**?**], [11], [12]. The nonlinear PDE stability is studied by transferring the common approaches based on finite dimension to infinite dimension. The theoretical proof has been given for the closed loop stability under a Proportional and a Proportional Integral (PI) controller with identical gains. In this paper, an analysis of the closed loop stability of the de Saint-Venant PDE is proposed with a general PI, using Multi-Models and the Internal Model Boundary Control (IMBC) structure. A variable elimination technique, as for finite-dimensional systems [3], [28], [31], has been used in order to solve a BOI (well known as Bilinear Matrix Inequality (BMI) in finite dimension) problem by the resolution of two LOI (well known as Linear Matrix Inequalities (LMI) in finite dimension).

The paper is organized as follows: firstly, the equations of de Saint-Venant are presented as well as the control problem. The Internal Model Boundary Control is explained and the physical constraints are given. In section II, the linearized models are defined around equilibrium sets as their insertion into the LOI formalism. Part III of the paper is dedicated to the design of the feedback gains by LOI & BOI techniques which ensures the stability of the system. A Proportional Integral (PI) controller is implemented based on two propositions: these are the major contributions of this paper. The last section IV is dedicated to simulations. Comparisons between the simulations with the PI controller (for which the proportional gain is equal to the integral gain, developed in previous works) and the new PI controller ($K_{int} \neq K_{pr}$ calculated using the LOI & BOI techniques) are realized. Simulations of a new channel are also implemented in the last section.

II. Problem statement about channel regulation

The control problem concerns the stabilization of the water flow rate and/or the water height around an equilibrium for a reach denoted by $(Z_e(x), Q_e(x))$.

A. A model of a reach

The channel is supposed to have a sufficient length L (from $x = 0$ to $x = L$) such that one can consider that the lateral movement is uniform. $Q(x,t)$, the water flow rate, and $Z(x,t)$, the height of the water, are the state variables. The nonlinear PDE of de Saint-Venant which describe the flow on the channel are [7], [16]:

$$\begin{cases} \partial_t Z = -\partial_x \frac{Q}{b}, \\ \partial_t Q = -\partial_x (\frac{Q^2}{bZ} + \frac{1}{2} gbZ^2) + gbZ(I - J), \end{cases} \tag{1}$$

$$y(t) = C[Z(x,t) \;\; Q(x,t)]^T \tag{2}$$

$$Z_0(x) = Z(x,0), Q_0(x) = Q(x,0) \tag{3}$$

$\forall x \in \Omega = (x_{up}, x_{do}) = (0, L)$, $t > 0$, $C : (L^2(0,L))^2 \to \mathbb{R}$. I is the slope, b is the channel width, g is the gravity constant. J is the friction slope from the formula of Manning-Strickler and R is the hydraulic radius. The considered boundary conditions $\forall x \in \Gamma = \partial\Omega$ are two underflow gates. The controlled variable is defined as follows:

$$Q(x,t) = U(t)\Psi(Z(x,t)) \tag{4}$$

with $\Psi(Z) = \kappa\sqrt{2g(Z_{up} - Z_{do})}$. Z_{up} is the water height at upstream of the gate, Z_{do} is the water height at downstream of the gate, κ is the product of the channel width with the water flow rate coefficient of the gate. The gates opening $U(t)$ is the control at upstream ($U_{up} = U_0$) and at downstream ($U_{do} = U_L$). The output variable is the water level at downstream, i.e. $Z(L)$.

B. A regulation model

The fluvial case, i.e. the subcritical case [22], is considered. Let $\xi(t) = (z(t) \quad q(t))^T$ be the linearized state variable, then the model around the equilibrium state $(Z_e(x) \quad Q_e(x))^T$ is:

$$\begin{aligned} \partial_t \xi(x,t) &= \mathcal{A}\xi(x,t) = A_1(x)\partial_x \xi(x,t) + A_2(x)\xi(x,t), && (5) \\ F_b \xi(t) &= B_b u(t) \text{ and } \xi(0,t) = \xi_0(t), && (6) \end{aligned}$$

where A_1 and A_2 are matrices of the space variable x. The linearized boundary conditions (6) are equivalent to:

$$q(x_{up},t) = U_{up,e}\partial_z \Psi(Z_e(x_{up},t))z(x_{up},t) + u_{up}(t)\Psi(Z_e(x_{up},t)), \quad (7)$$

$$q(x_{do},t) = U_{do,e}\partial_z \Psi(Z_e(x_{do},t))z(x_{do},t) + u_{do}(t)\Psi(Z_e(x_{do},t)), \quad (8)$$

where $U_{up,e}$ and $U_{do,e}$ are the opening gates for the upstream and downstream (respectively) at the equilibrium and $u_{up}(t)$, $u_{do}(t)$ are the variations of these opening gates to be controlled.
The control problem is to find the variations of $u_{up}(t)$ at extremity $x = x_{up}$ and $u_{do}(t)$ at extremity $x = x_{do}$ of the reach such that the downstream water level, $Z(x_{do},t) = Z(L,t)$ (measured variable) tracks a reference signal $r(t)$. The reference signal $r(t)$ is chosen for all cases either constant or non-persistent (a stable step answer of a non-oscillatory system).
In this paper, the control scheme based on the Internal Model Boundary Control (IMBC) is adopted [10]. This control strategy integrates the process model in real time and allows to regulate the water height in all the points of the channel by taking into account the error between the linearized model and the real system (or the nonlinear model for the simulations).

C. Open-loop system stability

Equation (5) describes the open loop system dynamics. In this representation, the state vector $\xi(x,t)$ is not explicitly linked with the boundary control. In order to design an output feedback and to study the closed-loop stability, a distribution operator D of control at the boundary is introduced [14][1], $D : \mathcal{C}^k([0,\infty], \mathbb{R}^n) \to (L^2(0,L))^2$. It is a bounded operator such that $Im(D) = Ker(\mathcal{A})$ and $Du \in D(\mathcal{A})$ and [10], [14], [32]:

$$\xi(x,t) = \varphi(x,t) + Du(t). \quad (9)$$

[1]Regularity coefficient is generally taken as $k = 2$

This operator is naturally null in the domain of $\mathcal{A}(x)$ as it is active only on the boundary of the domain. This change of variables allows to get a Kalman representation of the system [1], [14], [32]:

$$\begin{aligned} \partial_t \varphi(x,t) &= \mathcal{A}(x)\varphi(x,t) - D\dot{u}, & (10)\\ \varphi(x,0) &= \varphi_0(x) = \xi_0(x) - Du(0), & (11)\\ y(t) &= C\varphi(x,t) + CDu(t). & (12) \end{aligned}$$

It has been proved that the open loop system (10-12) is exponentially stable [10], as the operator of the linearized system in infinite dimension generates an exponentially stable C_0-semigroup. Moreover, under a PI-control $u(t) = \alpha_i K_i \int \varepsilon(s)ds + \alpha_p K_p \varepsilon(t) \in U = \mathbb{R}^n$, $u \in C^k([0,\infty],U)$, the stability of the closed-loop nonlinear system is ensured under some specific conditions on the gain synthesis. They are deduced from the properties of the IMBC structure and from the stability of the closed-loop linearized system.

For example, for the tuned gains of the PI-control the stability conditions are ensured if:

$$0 \leqslant \alpha_i < \alpha_{i,max} = \min_{\lambda\in\Gamma}(a\|R(\lambda;\mathcal{A}_e)\| + 1)^{-1}, \quad (13)$$

$$0 \leqslant \alpha_p < \alpha_{p,max} = (\sup_{\lambda\in\Gamma} a\|R(\lambda;\mathcal{A})\|)^{-1}, \quad (14)$$

where $\mathcal{A}_e$ is a part of the series development of the closed loop operator [10], $R(\lambda;K)$ is the resolvent operator of K, and a is a constant which depends on $\mathcal{A}_e$.

These theoretical results have been corroborated by simulations and experimentations [10]. Those experimentations have shown the limitations due to the linearization around an equilibrium state. A first attempt with a Multi-Models approach has been realized with success. However, it was not optimal and no theoretical proof has been given. The first approach by an integral control [12] had been extended to a PI control in [11], but the proportional and the integral gains were equal. The aim of this paper is to extend the previous results in infinite dimension with the proportional gain different from the integral gain. The theoretical proof realized in finite dimension with $K_{int} \neq K_{pr}$ [28] is developed to infinite dimensional systems. In order to control the water level over a wide operating range, a set of models is considered around judicious operating regimes: a control is synthesized and activated on the intervals when the system gets through the intervals.

D. A Multi-Models representation of de Saint-Venant's Equation

The Multi-Models representation [28], [12] of de Saint-Venant's PDE around N operating points is defined by the following equations:

$$\begin{aligned} \partial_t \xi(x,t) &= \sum_{i=1}^{N} \mu_i(\zeta(t)) \mathcal{A}_i(x) \xi(x,t) \text{ with } \mathcal{A}_i(x) = A_{1,i}(x)\partial_x + A_{2,i}(x), \\ \xi_0(x) &= \xi(x,0), \end{aligned} \tag{15}$$

where $\mathcal{A}_i(x)$ is the operator which corresponds to the i^{th} equilibrium state. $\zeta(t)$ is a function which depends on some decision variables directly linked with the measurable state variables and eventually to the input. $\mu_i(\zeta(t))$ are the weighting functions which activate the control law in function of the output of the process Z_L. They belong to a convex set such that

$$\sum_{i}^{N} \mu_i(\zeta(t)) = 1 \text{ and } \mu_i(\zeta(t)) \geq 0.$$

In the following section, the control law synthesis by LOI techniques is considered.

III. Study of the closed-loop system stability by LOI

In this part, the closed loop structure is studied under a proportional integral feedback. Recall that the aim is to control the water height over all the operating range, so the output $y(t)$ is not the variations around an equilibrium but the total water height. To this end, the output $y(t)$ is modified:

$$y(t) = C\xi(x,t) + Eq(x,t),$$

where $Eq(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t))(Z_{e,i}(x,t) \;\; Q_{e,i})^T$ is the equilibrium state and for this paper $CEq(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t)) Z_{e,i}(L)$ as the aim is to regulate the water level at $x = L$.

A. Closed-loop structure for a proportional integral feedback

Let K_{int} and K_{pr} be the integral and proportional gains respectively. It follows that [10]:

$$u(t) = K_{int} \int [r(\tau) - y(\tau)]d\tau + K_{pr}[r(t) - y(t)], \tag{16}$$

where $r(t)$ is the physical water level wanted (not the variations). So by using (9) one gets:

$$y(t) = C\varphi(x,t) + CEq(x,t) + CDu(t), \tag{17}$$

and by replacing $y(t)$ into the equation (16), it becomes:

$$\begin{aligned} u(t) &= K_{int} \int [r(\tau) - (C\varphi(x,\tau) + CEq(x,\tau) + CDu(\tau))]d\tau \\ &\quad + K_{pr}[r(t) - (C\varphi(x,t) + CEq(x,t) + CDu(t))] \end{aligned} \tag{18}$$

In each local model, $Eq(x,t)$ is a piecewise function ($\dot{E}q(x,t) = 0$)[2]. This is also the case of $r(t)$. So, $\dot{u}$ can be simplified to:

$$\dot{u}(t) = K_{int}[r(t) - C\varphi(x,t) - CEq(x,t) - CDu(t)] - K_{pr} C\mathcal{A}_i(x)\varphi(x,t) \tag{19}$$

By replacing $\dot{u}$ into the equation (10) and with $\tilde{K}_{int} = D\ K_{int}$, $\tilde{K}_{pr} = D\ K_{pr}$, the expression of the closed-loop system can be expressed as follows:

$$\begin{aligned} \partial_t \varphi(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t)) \big[(\mathcal{A}_i(x) + \tilde{K}_{int}C + \tilde{K}_{pr} C\mathcal{A}_i(x))\varphi(x,t) \\ + \tilde{K}_{int}\big(CDu(t) + CEq(x,t) - r(t)\big)\big] = \sum_{i=1}^{N} \mathcal{M}_i(x,t). \end{aligned} \tag{20}$$

The stability conditions are ensured by using a quadratic Lyapunov function [29] in order to guarantee the convergence of the water height to the reference $r(t)$ over the widest operating range.

B. *Stability study with a Lyapunov function*

Let us consider:

$$V(\varphi(x,t),t) = \langle \varphi(x,t), P\varphi(x,t) \rangle, \tag{21}$$

where $\langle .,. \rangle$ is the considered inner product. The Multi-Models representation of the linearized PDE of de Saint-Venant defined by equation (20) is asymptotically stable if there exists an operator $P > 0$, such that:

$$\langle \dot{\varphi}, P\varphi \rangle + \langle \varphi, P\dot{\varphi} \rangle = -\langle \varphi, \varphi \rangle. \tag{22}$$

The main difference here between the stability result in finite and infinite dimension [12], lies in the *inequality* of the Lyapunov function for finite dimensional systems and *equality* for infinite ones (22). This equality complexity can be removed in some cases; as for example for operators

[2] The following notation is considered: $\partial_t \phi = \dot{\phi}$ whatever the function ϕ.

with compact resolvent [5], [10], [33] or [30]. In this case, the same inequality from finite dimension is a sufficient and necessary condition for the infinite dimensional case; it needs to satisfy the spectral growth assumption [33], [14]. Moreover, for the equations of de Saint-Venant, it has been shown that the operator has a compact resolvent [10] so it satisfies the spectral growth assumption. Then, by taking into account (20)-(22), one has to prove the following inequality:

$$\langle \mathcal{M}_i, P\varphi \rangle + \langle \varphi, P\mathcal{M}_i \rangle \ < 0, \tag{23}$$

where $\mathcal{M}_i$ is defined in (20).

The development of the inequality (23) leads us to consider an inequality for each local system of index i such that[3]:

$$\begin{aligned} &\langle [\mathcal{A}_i + \tilde{K}_{int}C + \tilde{K}_{pr}C\mathcal{A}_i]\varphi(.,t), P\varphi(.,t)\rangle + \langle \tilde{K}_{int}[CDu(t) + CEq(.,t) - r(t)], P\varphi(.,t)\rangle \\ &+\langle \varphi(.,t), P[\mathcal{A}_i + \tilde{K}_{int}C + \tilde{K}_{pr}C\mathcal{A}_i]\varphi(.,t)\rangle + \langle \varphi(.,t), P\tilde{K}_{int}[CDu(t) + CEq(.,t) - r(t)]\rangle \ < 0. \end{aligned} \tag{24}$$

In the inequality (24), which defines the stability condition of the system $\forall i$, the control parameter u appears; this is a difficulty for the gain synthesis: $\tilde{K}_{int}$, $\tilde{K}_{pr}$.

A first approach was made in [11] with $K_{pr} = K_{int}$. In this paper, the previous results are improved as K_{int} is considered different from K_{pr}. It has been proved that a good choice of K_{int} and K_{pr} based on semigroup theory is $K_{int} = -\alpha_i[CD]^\dagger$ and $K_{pr} = \alpha_p[CD]^\dagger$ (where † stands for the right pseudo-inverse) in [10]. α_i and α_p are defined in (13)-(14). So, one can assume that $\exists \beta \in \mathbb{R}$ such that $K_{pr} = \beta K_{int}$, i.e. $\tilde{K}_{pr} = \beta\tilde{K}_{int}$. Then, the equation (24) becomes:

$$\begin{aligned} &\langle [\mathcal{A}_i + \tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i]\varphi(.,t), P\varphi(.,t)\rangle + \langle \tilde{K}_{int}[CDu(t) + CEq(.,t) - r(t)], P\varphi(.,t)\rangle \\ &+\langle \varphi(.,t), P[\mathcal{A}_i + \tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i]\varphi(.,t)\rangle + \langle \varphi(.,t), P\tilde{K}_{int}[CDu(t) + CEq(.,t) - r(t)]\rangle < 0 \end{aligned} \tag{25}$$

Note that the open-loop system (5)-(8) is exponentially stable as the closed loop one under a PI-control, with gains correctly tuned [10] for a time t well chosen. So, one can assume that $\exists k > 0$, such that:

$$\mid C\xi(x,t) + CEq(x,t) - r(t) \mid \leq k \mid C\varphi(x,t) \mid \tag{26}$$

and with $\alpha = (k+1)\varepsilon_{(\varphi^T P\tilde{K}_{int}C\varphi)}$ [12]:

$$\langle \varphi, P\tilde{K}_{int}(CDu(x,t) + CEq(x,t) - r(t))\rangle \ \leq \ \langle \varphi, \alpha P\tilde{K}_{int}C\varphi\rangle \tag{27}$$

[3]Due to the lack of space, (x,t) is replaced by $(.,t)$.

For finite dimensional systems, a stability study has been given in our paper [28], by using well-known linear techniques, but not developed for infinite dimensional systems. Here, the main contribution consists in a tool developed using the semigroup theory.

Proposition 1: Let Z be a Hilbert space and let G, U, V, X four linear operators on Z such that $G : D(G) \subset Z \to D(G)$. The domains of U, V and X are densely defined on $D(G)$, the domain of G. X is a self-adjoint operator such that $\|X\| \leq 2\sigma$.
If $\exists \sigma \in \mathbb{R} \backslash \{0\}$ which satisfy, $\forall \phi,\ \varphi \in D(G)$

$$\langle G\varphi + U^*\phi, \varphi\rangle + \langle U\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{28a}$$

$$\langle G\varphi + V^*\phi, \varphi\rangle + \langle V\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{28b}$$

then, the following inequality is also satisfied:

$$\langle G\varphi, \varphi\rangle + \langle U^*XV\varphi, \varphi\rangle + \langle V^*XU\varphi, \varphi\rangle \ < 0 \tag{40}$$

Proof: See appendix. ■

The following proposition extends our results [28] to infinite dimensional systems.

Proposition 2: If there exists a self-adjoint operator P, matrices W_{int} and W_{pr}, scalars $\sigma, \gamma \in \mathbb{R}$, such that the inequalities (28) are satisfied with $G = \mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*$, $U^* = W_{int}$, $V = C\mathcal{A}_i$, then one gets the following inequalities:

$$\langle (\mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*)\,\varphi + W_{int}\phi, \varphi\rangle + \langle W_{int}^*\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{41a}$$

$$\langle (\mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*)\,\varphi + (C\mathcal{A}_i)^*\,\phi, \varphi\rangle + \langle C\mathcal{A}_i\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0 \tag{41b}$$

and the closed-loop system (20) under the PI control law (16) is stable. ■

Proof: Let us consider (21) with the closed-loop system (20) under a PI control law (16). By considering the inequality (23), we can obtain (25) with $K_{pr} = \beta K_{int}$. Now, let us assume that the inequality (27) holds true, then (25) becomes:

$$\langle \left(P[\mathcal{A}_i + \gamma\tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i] \right)\varphi, \varphi\rangle + \langle \varphi, \left(P[\mathcal{A}_i + \gamma\tilde{K}_{int}C + \beta\tilde{K}_{int}C\mathcal{A}_i] \right)\varphi\rangle < 0 \tag{42}$$

with $\gamma = 1+\alpha$, $\tilde{K}_{int} = P^{-1}W_{int}$, $\tilde{K}_{pr} = \beta\tilde{K}_{int}$. The inequality (42) has two variables: β and W_{int} that lead to a BOI problem. By using proposition 1, the BOI problem can be solved as two LOI problem. The inequality (42) is equivalent to (40) with $G = \mathcal{A}_i^*P + P\mathcal{A}_i + \gamma W_{int}C + \gamma C^*W_{int}^*$, $U^* = W_{int}$, $V = C\mathcal{A}_i$, $X = \beta Id$. Proposition 1 allows to conclude that if the inequalities (41) are satisfied, the Lyapunov inequality is also true. Thus the system (15) under PI control is stable. ■

Remark 3: The inequalities (41) seem to be linked with the stability of each submodel in infinite dimension, with a Lyapunov Input-to-state stability (ISS) function [24], [6].

The aim of section IV is to compare the simulated curves obtained with this method and the ones obtained on the experimental benchmark [10] and the simulations with the gains $\tilde{K}_{int} = \tilde{K}_{pr}$ [11], [12].

IV. Simulation results

Two benchmarks are used for the simulations: the micro-channel of Valence (France) and the channel of Gignac (France). The simulations are based on a Chang and Cooper scheme, for more details see [9], [12].

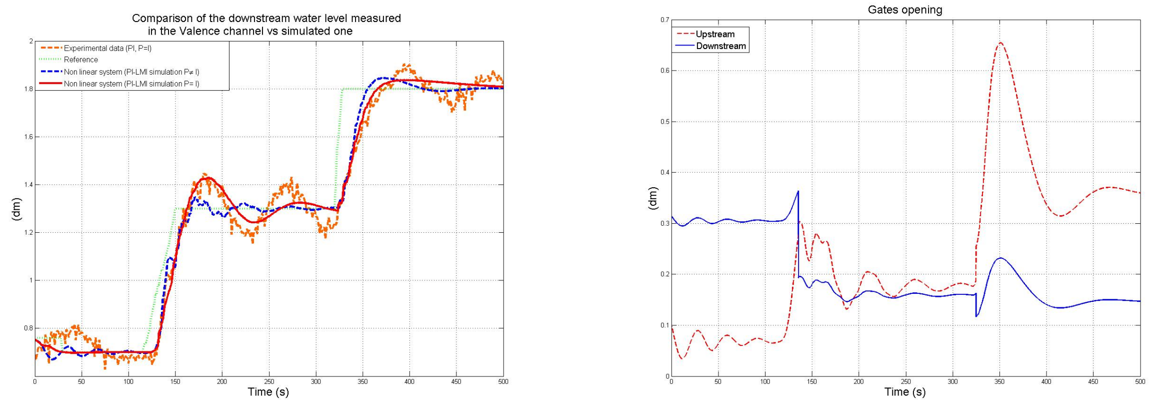
For both applications, the weighting function $\mu_i(\zeta(t))$ is equal to 1 if the output's height is included into the validity domain of the model and 0 in the other case for each operating state. The output of the system is the decision variable. The parameter $\zeta(t)$ is a function of it. Both coefficients β and σ of the proposition 1 are negative in both cases simulated, so the condition $\|X\| \leq -2\sigma$ is always satisfied whatever $X = \beta Id$. Here, the inequalities (41) are solved after discretization by LMI as in [28].

A. The micro-channel of Valence

The equilibrium profiles have been chosen such that the calculated control law from the local models can be efficient over all the operating range of the water height [10]. Notice that it has been experimentally verified that a local model is valid around $\pm 20\%$ of an equilibrium profile. In order to assign references which are included between $0.06\ m$ and $0.2\ m$, the operating points at $x = 0$ are given in the Table I.

The following simulation (Figure (1.a)) compares PI controllers: one with gains $K_{int} = K_{pr}$ [11] and the new one with gains $K_{int} \neq K_{pr}$. The Figure (1.b) represents the dynamic evolution of the gates opening of the simulated system.

It will be observed in Figure (1.a) that the water height convergence with the new PI controller is better than the one obtained with $K_{int} = K_{pr}$ [11] and the overshoot is less too. It is also



a) Comparison of the downstream water level b) Gates opening

Fig. 1. Valence channel simulation

better than the experimental PI which has been implemented in [10].

TABLE I

INITIAL SET POINTS FOR THE SIMULATION OF THE CHANNELS OF VALENCE AND GIGNAC

	$z_{e1}(x=0)$	$z_{e2}(x=0)$	$z_{e3}(x=0)$
Valence	0.06 m	0.1 m	0.16 m
Gignac	1.09 m	1.46 m	1.68 m

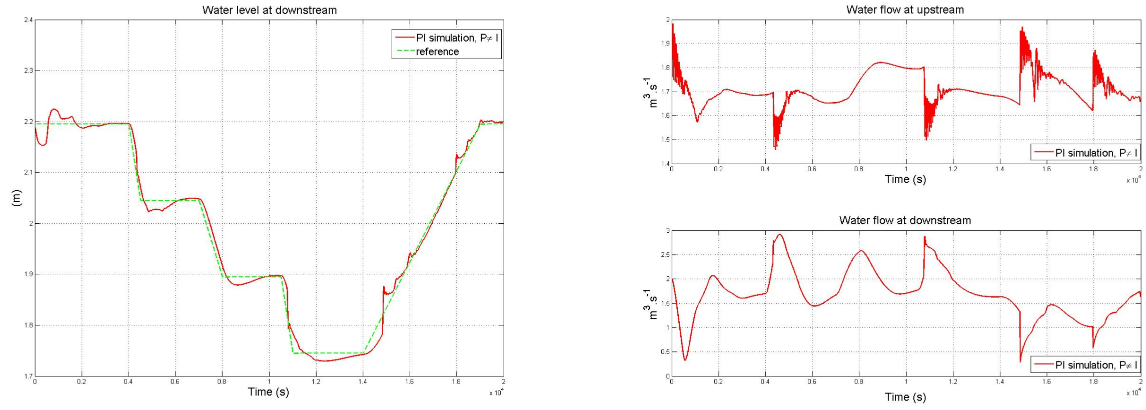
B. The channel of Gignac

In this paper, we have also studied a channel which is located in Gignac (France). The following set of parameters of this channel is considered: $L = 2272\ m$ is the length of the channel, $b = 3\ m$ is the width of the channel, $N = 40$ is the number of the discretizated points, Z_L is the water height to regulate such that $1.7\ m < Z < 2.5\ m$.

Figure (2.a) shows that the output converges to the reference over a wide operating range. In Fig. (2.b), the water flows at the upstream and downstream stay in physical proportion, which is an important practical point.

V. CONCLUSION

First attempts of a Multi-Models approach on irrigation channels control, through an IMBC structure, have been realized some years ago [10]. Good experimental results were obtained



a) Comparison of the downstream water level b) Water flows

Fig. 2. Gignac channel simulation

which showed promising results but a theoretical basis was lacking. The first theoretical results in order to design the feedback gain through LMI have been realized in the case of an Integral controller in [12]. Preliminary results of a PI controller in a particular case ($K_{int} = K_{pr}$) have been published in [11] for infinite dimensional systems and with $K_{int} \neq K_{pr}$ in [28] for finite dimensional systems. In the present paper, the authors take into account the more general case of PI controller with $K_{int} \neq K_{pr}$ for infinite dimensional systems. They synthesize the new PI controller feedback gains by solving a BOI problem. Simulations show a better performance than the previous results, through this new PI feedback controller designed by BMI $\&$ LMI [12]. The forthcoming experimentation should emphasize the improvements of the new PI controller with weighting functions, non-piecewise constant ($\mu_i(\zeta(t)) \in [0, 1]$).

VI. Proof of the Proposition 1

Proposition 4: Let Z be a Hilbert space and let G, U, V, X four linear operators on Z such that $G : D(G) \subset Z \to D(G)$. U, V and X are densely defined on $D(G)$, the domain of G. X is a self-adjoint operator such that $\|X\| \leq 2\sigma$. If $\exists \sigma \in \mathbb{R}$ which satisfy, $\forall \phi, \ \varphi \in D(G)$

$$\langle G\varphi + U^*\phi, \varphi\rangle + \langle U\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0, \tag{43a}$$

$$\langle G\varphi + V^*\phi, \varphi\rangle + \langle V\varphi + \sigma^{-1}\phi, \phi\rangle \ < 0, \tag{43b}$$

then, the following inequality is also satisfied:

$$\langle G\varphi, \varphi\rangle + \langle U^{*}XV\varphi, \varphi\rangle + \langle V^{*}XU\varphi, \varphi\rangle \ < 0. \tag{44}$$

Proof: The objective is to prove this equation (44):

$$\langle G\varphi, \varphi\rangle + \langle U^{*}XV\varphi, \varphi\rangle + \langle V^{*}XU\varphi, \varphi\rangle < 0$$

$$\Rightarrow \langle G\varphi, \varphi\rangle + \langle XV\varphi, U\varphi\rangle + \langle XU\varphi, V\varphi\rangle < 0 \tag{45}$$

$$\langle G\varphi, \varphi\rangle + \langle V\varphi, XU\varphi\rangle + \langle U\varphi, XV\varphi\rangle < 0 \tag{46}$$

Developing (43), one can obtain the following inequalities:

$$\begin{cases} \langle G\varphi, \varphi\rangle + \langle U^{*}\phi, \varphi\rangle + \langle U\varphi, \phi\rangle + \sigma^{-1}\|\phi\|^{2} < 0, \\ \langle G\varphi, \varphi\rangle + \langle V^{*}\phi, \varphi\rangle + \langle V\varphi, \phi\rangle + \sigma^{-1}\|\phi\|^{2} < 0. \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle G\varphi, \varphi\rangle + \langle \phi, U\varphi\rangle + \langle U\varphi, \phi\rangle + \sigma^{-1}\|\phi\|^{2} < 0 \\ \langle G\varphi, \varphi\rangle + \langle \phi, V\varphi\rangle + \langle V\varphi, \phi\rangle + \sigma^{-1}\|\phi\|^{2} < 0 \end{cases} \tag{47}$$

Let consider firstly that σ is positive, $\phi = XV\varphi$ and $\phi = XU\varphi$ in equations (47), one gets:

$$\begin{cases} \langle G\varphi, \varphi\rangle + \langle XV\varphi, U\varphi\rangle + \langle U\varphi, XV\varphi\rangle + \sigma^{-1}\|XV\varphi\|^{2} < 0, \\ \langle G\varphi, \varphi\rangle + \langle XU\varphi, V\varphi\rangle + \langle V\varphi, XU\varphi\rangle + \sigma^{-1}\|XU\varphi\|^{2} < 0. \end{cases} \tag{48}$$

Summing those both inequalities of (48), one obtains:

$$2\left\{\langle G\varphi, \varphi\rangle + \langle XV\phi, U\varphi\rangle + \langle U\varphi, XV\phi\rangle\right\} + \sigma^{-1}\left(\|XV\varphi\|^{2} + \|XU\varphi\|^{2}\right) < 0. \tag{49}$$

With $\sigma^{-1} > 0$, it is equivalent to:

$$\Leftrightarrow \langle G\varphi, \varphi\rangle + \langle V\varphi, XU\varphi\rangle + \langle U\varphi, XV\varphi\rangle < 0. \tag{50}$$

So the inequality (46) and also the inequality (44) have been proved, the proposition is verified. **Let consider now that $\sigma^{-1} < 0$**, four constants $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$, and the inequalities (47), with this time $\phi = \alpha U\varphi + \beta XV\varphi$ and $\phi = \gamma V\varphi + \delta XU\varphi$. One can obtained the following inequalities:

$$\begin{cases} \langle G\varphi,\varphi\rangle + \langle \phi, U\varphi\rangle + \langle U\varphi,\phi\rangle + \sigma^{-1}\|\phi\|^2 < 0, \\ \langle G\varphi,\varphi\rangle + \langle \phi, V\varphi\rangle + \langle V\varphi,\phi\rangle + \sigma^{-1}\|\phi\|^2 < 0 \end{cases} \tag{51}$$

$$\Rightarrow \begin{cases} \langle G\varphi,\varphi\rangle + \langle \beta XV\varphi, U\varphi\rangle + \langle U\varphi, \beta XV\varphi\rangle \\ \quad +\langle \alpha U\varphi, U\varphi\rangle + \langle U\varphi, \alpha U\varphi\rangle + \sigma^{-1}\|\alpha U\varphi + \beta XV\varphi\|^2 < 0, \\ \langle G\varphi,\varphi\rangle + \langle \delta XU\varphi, V\varphi\rangle + \langle V\varphi, \delta XU\varphi\rangle \\ \quad +\langle \gamma V\varphi, V\varphi\rangle + \langle V\varphi, \gamma V\varphi\rangle + \sigma^{-1}\|\gamma V\varphi + \delta XU\varphi\|^2 < 0. \end{cases}$$

Let sum both equations, then one finds:

$$\begin{aligned} \Rightarrow \quad & 2\langle G\varphi,\varphi\rangle + (\beta+\delta+\sigma^{-1}(\alpha\beta+\gamma\delta))\,[\langle XV\varphi,U\varphi\rangle + \langle U\varphi, XV\varphi\rangle] + \sigma^{-1}\beta^2\langle XV\varphi, XV\varphi\rangle \\ & +(2\alpha+\sigma^{-1}\alpha^2)\langle U\varphi,U\varphi\rangle + (2\gamma+\sigma^{-1}\gamma^2)\langle V\varphi,V\varphi\rangle + \sigma^{-1}\delta^2\langle XU\varphi, XU\varphi\rangle \\ \leq \quad & 2\langle G\varphi,\varphi\rangle + (\beta+\delta+\sigma^{-1}(\alpha\beta+\gamma\delta))\,[\langle XV\varphi,U\varphi\rangle + \langle U\varphi, XV\varphi\rangle] \\ + \quad & (2\alpha+\sigma^{-1}[\alpha^2+\|X\|^2\delta^2])\langle U\varphi,U\varphi\rangle + (2\gamma+\sigma^{-1}[\gamma^2+\|X\|^2\beta^2])\langle V\varphi,V\varphi\rangle < 0. \end{aligned} \tag{52}$$

The aim is to defined the constants $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ such that $\{(52)\Rightarrow (44)\}$ is satisfied $\forall$ $\sigma^{-1} < 0$. Three constraints appear:

1) $\beta+\delta+\sigma^{-1}(\alpha\beta+\gamma\delta) = 2$,
2) $(2\alpha+\sigma^{-1}[\alpha^2+\delta^2\|X\|^2]) > 0$,
3) $(2\gamma+\sigma^{-1}[\gamma^2+\beta^2\|X\|^2]) > 0$.

If they are satisfied then $\langle G\varphi,\varphi\rangle + \langle XV\varphi,U\varphi\rangle + \langle U\varphi, XV\varphi\rangle < 0$.

Consider the first constraint and let define $\alpha = \gamma = -\sigma\, m$, where $m \in \mathbb{R}$, then the first equality implies that $\beta+\delta = \frac{2}{1-m}\ m \neq 1$.

As β and δ have the same role, we define for $m \neq 1$ $\beta = \delta = \frac{1}{1-m}$ then

$$(2\alpha+\sigma^{-1}[\alpha^2+\delta^2\|X\|^2]) \quad = \quad -2\sigma * m + \sigma m^2 + \frac{\sigma^{-1}\|X\|^2}{(1-m)^2}, \tag{53}$$

$$(2\alpha+\sigma^{-1}[\alpha^2+\delta^2\|X\|^2]) > 0 \quad \Leftrightarrow \quad \sigma^2 m(2-m)(1-m)^2 > \|X\|^2 \text{ iff } 0 < m < 2 \tag{54}$$

Idem for the third inequality, as the coefficients are equals.

So, for every $\sigma < 0$, $\exists 0 < m < 2,\ m \in \mathbb{R}\backslash\{1\}$ and an operator X self-adjoint such that

$$\sigma\sqrt{m(2-m)}(1-m) < -\|X\|,$$

and the three constraints above

are satisfied with $\alpha = \gamma = -\sigma\, m$ and $\beta = \delta = \frac{1}{1-m}$. As it is true for all m, let take $m = 1 - \sqrt{2}/2$, then $\sqrt{m(2-m)}(1-m) = 0.5$ and one gets $\sigma < -2\|X\|$. Then, the inequality (52) is realized and (46)⇔(44) is too. The proposition is proved. ■

Remark 5: The choice of m is arbitrary, and one can consider another one ($m \neq 1,\ 0 < m < 2$). Here we have chosen the critical case i.e. the minimum value of the inverse function of $m(2-m)(1-m)^2$.

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A Proportional Integral Feedback for Open Channels Control trough LMI Design

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Abstract: This paper deals with the controller design for systems described by Partial Differential nonlinear Equation (PDE) of Saint-Venant. The proposed approach is based on Multi-Models concept which takes into account Linear Time Invariant models defined around a set of operating points. This method allows to describe the dynamic of this nonlinear system over a wide operating range. By the means of an Internal Model Boundary Control (IMBC), the design of a Proportional Integral (PI) feedback is performed through Linear Matrix Inequality (LMI). The method is applied to simulation and also compared to previous experimentations on a micro-channel, illustrating the new theoretical results developed in the paper.

1. INTRODUCTION

Regulation of open channels represents an economic and environment interest and many research are done in this area. Indeed, water losses in open channels, due to inefficient management and control, may be large. In order to deliver water, it is important to ensure that the water level and the flow rate in the open channel remain at certain values [21]. The difficulty of this control system is that only the gates positions are able to meet performance specifications so a specific design with boundary control laws satisfying the control objectives is required.

This problem has been previously considered in the literature using a wide variety of technics see [20, 29]. Some of them take into account the uncertainties and apply robust control approach (see [16, 17] e.g.). Studying directly the nonlinear dynamics is also possible as in [30, 15, 9]. Recent approaches consider the distributed feature of the system. Using the Riemann coordinates approach on the Saint-Venant equations, stability results are given in [13] for a system of two conservation laws, and for system of larger dimension. Lyapunov technics have been used in [5, 9].

In practice, process industries as chemical, water treatment processes are characterized by complex processes which often operate in multiple operating regimes. It is often difficult to obtain nonlinear models that accurately describe plants in all regimes. Also, considerable effort is required for development of nonlinear models. An attractive alternative to nonlinear technique is to use a multi-linear model strategy. Multi-linear models methods are based on the partitioning of the operating range of a system into separate regions and applying local linear models to each region [22]. The Multi-Models structure is well adapted for nonlinear systems because this structure allows to determine a set of linear models defined around some predefined operating points. Each local model (submodel) is defined as a LTI dynamic system defined for a specific operating point. The Multi-Models philosophy is based on weighting functions which ensure the transition between the different locals models. These functions represent the degree of validity of each local models and it depends on the system inputs and outputs which vary with time. The multi-model approach has been often used in recent years for modelling and control of nonlinear systems [27, 24, 2] and for fault diagnosis [3, 11, 26]. In the Multi-Models concept, some authors speak about gain scheduling strategy which is well detailed in [14] or for interpolated controllers or switching controllers [23, 1].

The use of Multi-Models representation for stability study of systems described by nonlinear PDE is not present in the literature. More generally, common approaches are based on a finite dimensional approximation of the nonlinear PDE and adaptive control. The stability for such systems is still an open problem. In this paper, an analysis of the stability of the nonlinear PDE of Saint-Venant is proposed by the use of the Multi-Models and IMBC [28] structures. A previous work from the authors has been published with the use of an Integral control. The goal of this paper is to extend the theory with the use of a PI Control. The stability in Multi-Models framework is often performed by Linear Matrix Inequality (LMI) due to the effectiveness for calculating a gain solution for multiple models [18, 25].

The paper is organized as follows: firstly, the Saint-Venant equations are presented as well as the control problem. The Internal Model Boundary Control is explained and the physical constraints are given. Secondly, the linearized systems are developed around equilibrium sets which depend on the space variable. Their insertion into the LMI formalism are also described into this second part. The third part of the paper is dedicated to the design of the feedback gain by LMI which ensures the stability of the system: a Proportional Integral (PI) controller is implemented using a LMI approach and the local stability of each systems. The last section is dedicated to the simulations and comparison with previous experimentations. Comparisons between initial experimental results using a PI-controller (done some years ago) and simulations with the new PI-controller using the LMI gain calculated through this paper, are realized.

2. PROBLEM STATEMENT ABOUT CHANNEL REGULATION

Let us consider the following class of water channels represented on the figure (1), i.e. a reach of an open channel delimited by underflow and/or overflow gates where:

- $Q(x,t)$ is the water flow rate,
- $Z(x,t)$ is the height of water channel,
- L is the length of the reach taken between the upstream $x=0$ and the downstream $x=L$,
- $U_0(t)$, $U_L(t)$ are the opening of the gates at upstream and downstream.

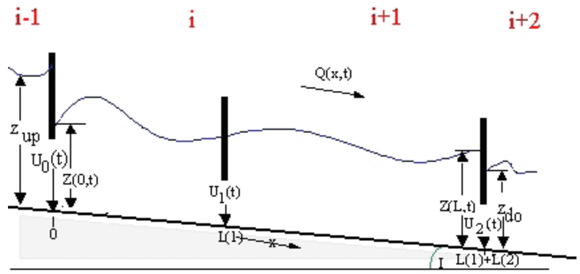

Fig. 1. Channel scheme: two underflow gates

The regulation problem concerns the stabilization of the water flow rate and/or the height of the water around an equilibrium for a reach denoted by $(z_e(x), q_e(x))$. A linear model with variable coefficients can be deduced from the nonlinear PDE, in order to describe the variation of the water level and flow on an open channel. Let recall these models of [6].

2.1 A model of a reach

We suppose that the channel has a sufficient length L such that we can consider that the lateral movement is uniform. Nonlinear PDE of de Saint-Venant which describe the channel are the following [12, 19]:

$$\partial_t Z = -\partial_x \frac{Q}{b}, \tag{1}$$

$$\partial_t Q = -\partial_x\left(\frac{Q^2}{bZ} + \frac{1}{2}gbZ^2\right) + gbZ(I-J), \tag{2}$$

$$Z_0(x) = Z(x,0), \qquad Q_0(x) = Q(x,0), \tag{3}$$

where I is the slope, b is the channel width, g is the gravity constant.

J is the friction slope from the formula of Manning-Strickler and R is the hydraulic radius. J and R are defined such that:

$$J = \frac{n^2 Q^2}{(bZ)^2 R^{4/3}}, \quad R = \frac{bZ}{b+2Z}. \tag{4}$$

The different limits conditions bring us to consider 2 control cases (mono or multi variable control). Here, the control of two underflow gates is considered.

Multi-variable control :

The equation of the upstream condition of the reach $(x = x_{up})$ is given by $Q(x_{up},t) = U_{up}(t)\Psi_1(Z(x_{up},t))$, with $\Psi_1(Z) = K_1\sqrt{2g(z_{up}-Z)}$. The other control at downstream of the reach, i.e. in $x = x_{do}$ (Fig. 1) is given by: $Q(x_{do},t) = U_{do}(t)\Psi_3(Z(x_{do},t))$, where $\Psi_3(Z) = K_2\sqrt{2g(Z-z_{do})}$ and $U_{do}(t)$ is the downstream control of the reach, z_{do} is the water height on downstream of the gate (cf. figure 1).

Remark 1. Upstream and downstream depend of the considered reach, it is the same thing for abscissa and gates.

2.2 A regulation model

An equilibrium point of the system verifies the following equations:

$$\partial_x q_e = 0$$
$$\partial_x z_e = gbz_e \frac{I + J_e + \frac{4}{3}J_e\frac{1}{1+2z_e/b}}{gbz_e - q_e^2/bz_e^2}, \tag{5}$$

Remark 2. The fluvial case is considered and it follows that:

$$z_e > \sqrt[3]{q_e^2/(gb^2)}. \tag{6}$$

Let denote that q_e is constant but that z_e depends of variable space. The linearized model around an equilibrium point $(z_e(t)\;\; q_e(t))^t$ is, with

$$\xi(x,t) = (z(x,t)\;\; q(x,t))^t, \forall t>0, x \in \Omega =]0, L[$$

the linearized state variables:

$$\begin{aligned}
&\partial_t \xi(x,t) = A_1(x)\partial_x \xi(x,t) + A_2(x)\xi(x,t) &(7)\\
&\xi(x,0) = \xi_0(x)\\
&q(x_{up}) = u_{up,e}\partial_z \Psi_1(z_e(x_{up}))z(x_{up}) + u_{up}\Psi_1(z_e(x_{up})),\\
&q(x_{do}) = u_{do,e}\partial_z \Psi_3(z_e(x_{do}))z(x_{do}) + u_{do}\Psi_3(z_e(x_{do}))
\end{aligned}$$

$\forall t > 0$, where $u_{up,e}$, $u_{do,e}$ are the openings gates for the upstream and downstream at the equilibrium and $u_{up}(t)$, $u_{do}(t)$ are the variations of these openings gates to be controlled. The matrices $A_1(x)$, $A_2(x)$ are given by:

$$A_1 = \begin{pmatrix} 0 & -a_1 \\ -a_2 & -a_3 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ a_4 & -a_5 \end{pmatrix}, \tag{8}$$

with $a_1(x) = 1/b$, $a_2(x) = gbz_e(x) - \frac{q_e^2}{bz_e^2(x)}$, $a_3(x) = \frac{2q_e}{bz_e(x)}$, $a_5(x) = \frac{2gbJ_e(x)z_e(x)}{q_e}$, $a_4(x) = gb(I + J_e(x) + \frac{\frac{4}{3}J_e(x)}{1+2z_e(x)/b})$.

The control problem is to find the variations of $u_{up}(t)$ at extremity $x = x_{up}$ and $u_{do}(t)$ at the extremity $x = x_{do}$ of the reach such that downstream water level, $z(x_{do},t) = z(L,t)$ (measured variables), track a reference signal $r(t)$. The reference signal $r(t)$ is chosen for all cases or constant or non-persistent (a stable step answer of a non-oscillatory system).

In this paper, the control scheme based on the Internal Model Boundary Control (IMBC) [8, 10] is adopted as illustrated on figure (2). This control strategy integrates the process model in real time and allows to regulate the water height in all the points of the channel by taking into account the error between the model and the system.

- M_f is the linear filtering model of finite dimension.

- M_r is the pursuit model which allows to set a dynamic in regards of the fixed instruction $r(t)$.

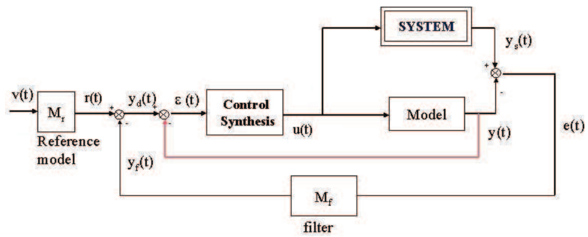

Fig. 2. IMBC structure: Internal Model Boundary Control

In order to control the water level over a wide operating range, we consider a set of models established around judicious operating points: each model is an approximation of the process in a small interval of the operating range and it should be activated to synthesize a control on this interval. The idea is to construct a set of predefined models in order to control the system over all the operating range.

2.3 A Multi-Models representation

The Multi-Models structure [26] allows to control the system over a wide operating range because it takes into account the different sub-models which can be activated under different operating regimes [22, 26]. The representation of Saint-Venant's PDE around N operating points by the Multi-Models approach is defined by the following equations:

$$\partial_t \xi(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t)) \mathcal{A}_i(x) \xi(x,t), \forall t > 0, x \in \Omega \quad (9)$$
$$\mathcal{A}_i(x) = A_{1,i}(x)\partial_x + A_{2,i}(x) \quad (10)$$
$$\xi_0(x) = \xi(x,0)$$
$$F_b \xi(t) = B_b u(t) \text{ on } \Gamma = \partial\Omega, \forall t > 0 \quad (11)$$

- $\mathcal{A}_i(x)$ is the operator which corresponds to the i^{th} equilibrium state.
- $\zeta(t)$ is a function depending of some decision variables directly linked with the mesurables states variables and eventually to the input.
- $\mu_i(\zeta(t))$ is the weighting functions which determines the sub-model for the control law synthesis depending of the output height of the process z_L.

The equation (9) describes the system dynamic in open loop. In this representation, the state vector $\xi(x,t)$ is not explicitly linked with the boundary control. In order to design an output feedback and to study the stability in closed loop, an operator D of distribution of the boundary control is introduced, it is a bounded operator such that $Du \in Ker(A)$, [8]:

$$\xi(x,t) = \varphi(x,t) + Du(t). \quad (12)$$

This operator is naturally null in the domain of $A(x)$ as it is active only on the boundary of the domain. This change of variables allows to get a Kalman representation of the system:

$$\partial_t \varphi(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t)) \left[\mathcal{A}_i(x)\varphi(x,t) - D\dot{u}\right] \quad (13)$$
$$\varphi(x,0) = \varphi_0(x) = \xi_0(x) - Du(0). \quad (14)$$

A Multi-Model approach can be developed and made possible the study of the stability by the second theory of Lyapunov.

In the following paragraph, we will focus on the synthesis of a control law by LMI technics. An output feedback is considered under a hypothesis of an integral control, so as to do a synthesis of a gain by LMI which ensures the stability of the system.

3. STABILITY STUDY BY LMI

In this part, the closed loop structure (Fig. 2) is studied under a proportional integral feedback. The pursuit model (M_r) and filtering model (M_f) are not considered.

3.1 Closed-loop structure for a Proportional-Integral feedback

For a control with an output feedback, K_i and K_p are defined as the gains, $u\ (t) \ = \ K_p\ \varepsilon(t) + K_i\ \int \varepsilon(t)dt$, it follows that [8]:

$$\varepsilon(t) = r(t) - y(t) \quad (15)$$
$$u(t) = K_p\ [r(t) - y(t)] + K_i \int [r(\tau) - y(\tau)]d\tau \quad (16)$$

with $y(x,t) = C\xi(x,t)$ and equation (12), one deduce:

$$y(x,t) = C\varphi(x,t) + CDu(t) \quad (17)$$

To clarify equations, the index i of the i-th equilibrium is omitted in the following (e.g. $\mathcal{A}$ replace $\mathcal{A}_i$). By replacing $y(x,t)$ into the control equation:

$$\begin{aligned} u &= K_p\ [r - C\varphi - CDu] + K_i \int [r - C\varphi - CDu]\, d\tau \\ \Rightarrow \dot{u} &= K_p\ [\dot{r} - C\dot{\varphi} - CD\dot{u}] + K_i\, [r - C\varphi - CDu] \\ &= K_p\ [\dot{r} - C\mathcal{A}\varphi] + K_i\, [r - C\varphi - CDu] \end{aligned}$$

using the change of variables (12) and the fact that $Im(D) \subset Ker(A)$ (See [8]). Then $\dot{u}$ is introduced into the equation (13) and, the closed-loop expression is then

$$\partial_t \varphi(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t)) \big[DK_i\,(CDu(t) - r(t)) - DK_p \dot{r}(t) + (\mathcal{A}_i(x) + DK_p\ C\mathcal{A}_i + DK_i C)\,\varphi(x,t)\big] \quad (18)$$

Let define:

$$\widetilde{K_{in}} = DK_i \qquad \widetilde{K_{pr}} = DK_p \quad (19)$$

The equation (18) can be written as

$$\partial_t \varphi = \sum_{i=1}^{N} \mu_i(\zeta) \Big[\Big(\mathcal{A}_i + \widetilde{K_{pr}}\ C\mathcal{A}_i + \widetilde{K_{in}} C\Big)\, \varphi + \widetilde{K_{in}}\,(CDu - r) - \widetilde{K_{pr}}\dot{r}\Big]. \quad (20)$$

The conditions which ensure the stability are ensured by using a quadratic Lyapunov function [25, 4], in order to guarantee the convergence of the water height to the reference $r(t)$, over the widest operating range.

3.2 Stability study with a quadratic Lyapunov function

Let us consider:

$$V(\varphi(x,t),t)=\varphi^T(x,t)P\varphi(x,t). \tag{21}$$

The Multi-Models representation of the linearized PDE of Saint-Venant defined by equation (20) is asymptotically stable if there exists a matrix $P>0$ such that [1]:

$$\dot V(\varphi,t)=\dot\varphi^T P\varphi+\varphi^T P\dot\varphi<0, \tag{22}$$

Then, it follows this inequality

$$\begin{aligned}
&\Bigg[\sum_{i=1}^{N}\mu_i(\zeta)\Big[\Big(\mathcal{A}_i+\widetilde{K_{pr}}\;C\mathcal{A}_i+\widetilde{K_{in}}C\Big)\varphi && (23)\\
&+\widetilde{K_{in}}\,(CDu-r)-\widetilde{K_{pr}}\dot r\Big]\Bigg]^T P\varphi \\
&+\varphi^T P\Bigg[\sum_{i=1}^{N}\mu_i(\zeta)\Big[\Big(\mathcal{A}_i+\widetilde{K_{pr}}\;C\mathcal{A}_i+\widetilde{K_{int}}C\Big)\varphi && (24)\\
&+\widetilde{K_{in}}\,(CDu-r)-\widetilde{K_{pr}}\dot r\Big]\Bigg]<0 && (25)
\end{aligned}$$

The development of this inequality leads us to consider an inequality for each i such that:

$$\begin{aligned}
&\varphi^T\left[\mathcal{A}_i+\widetilde{K_{pr}}\;C\mathcal{A}_i+\widetilde{K_{in}}C\right]^T P\varphi\\
&+\varphi^T P\left[\mathcal{A}_i+\widetilde{K_{pr}}\;C\mathcal{A}_i+\widetilde{K_{in}}C\right]\varphi\\
&+[\widetilde{K_{in}}(CDu-r)]^T P\varphi+\varphi^T P[\widetilde{K_{in}}(CDu-r)]\\
&-(\widetilde{K_{pr}}\dot r)^T P\varphi-\varphi^T P(\widetilde{K_{pr}}\dot r)<0
\end{aligned} \tag{26}$$

Remark 3. In a first approach, the particular case $K_{prop}=K_{int}$ is taken and so one get $\widetilde{K_{pr}}=\widetilde{K_{in}}=\tilde K$. Then equation (26) become:

$$\begin{aligned}
&\varphi^T\left[\mathcal{A}_i+\tilde K\;C\mathcal{A}_i+\tilde KC\right]^T P\varphi+[\tilde K(CDu-r)]^T P\varphi\\
&+\varphi^T P\left[\mathcal{A}_i+\tilde K\;C\mathcal{A}_i+\tilde KC\right]\varphi+\varphi^T P[\tilde K(CDu-r)]\\
&-(\tilde K\dot r)^T P\varphi-\varphi^T P(\tilde K\dot r)<0
\end{aligned} \tag{27}$$

In the inequality (27), which defines the stability condition of the system, the control parameter u appears in this inequality and it is a difficulty for the design of the gain $\tilde K$. Let us consider the following equality deduced from (17):

$$CDu(t)-r(t)=C\xi(x,t)-r(t)-C\varphi(x,t) \tag{28}$$

Proposition 1. If there exists a matrix P positive definite, a matrix W and a scalar α such that the following statements hold true:

a) $\varphi^T P\widetilde K\,(CDu(t)-r(t))\le\alpha\varphi^T P\widetilde KC\varphi,$ (29)
b) $\mathcal{A}_i^T P+P\mathcal{A}_i+W\mathcal{C}_i+\mathcal{C}_i^T W^T<0,$ with $\widetilde K=P^{-1}W$ and with $\mathcal{C}_i=\;C\mathcal{A}_i+(1+\alpha)C$
c) r is constant by piecewise,
then the system (13) with a proportional integral control input (16) is stable. ■

[1] We suppose that $\partial_t\psi=\dot\psi$ whatever the function ψ.

Proof 1. Only a sketch of the proof is given here. For the integral case, one can found it in [7].

Let consider the quadratic Lyapunov function

$$V(\varphi(x,t),t)=\varphi^T(x,t)P\varphi(x,t)$$

then one can wrote $\dot V(t)<0$ such that (27) can be upper bounded. Indeed, the use of inequality (29) implies that (the same is done for the transposed expression)

$$\begin{aligned}
&\varphi^T P\left[\mathcal{A}_i+\tilde K\;C\mathcal{A}_i+\tilde KC\right]\varphi+\varphi^T P[\tilde K(CDu-r)]\\
&-\varphi^T P(\tilde K\dot r)<\varphi^T P\left[\mathcal{A}_i+\tilde K\;C\mathcal{A}_i+(1+\alpha)\tilde KC\right]\varphi\\
&-\varphi^T P(\tilde K\dot r)<\varphi^T P\left[\mathcal{A}_i+\tilde K\mathcal{C}_i\right]\\
&=\varphi^T\left[P\mathcal{A}_i+W\mathcal{C}_i\right]\varphi<0
\end{aligned} \tag{30}$$

with $\mathcal{C}_i=\;C\mathcal{A}_i+(1+\alpha)C$, $W=P\tilde K$ and using the condition a) firstly, the third c) and the second condition b) of the proposition.

The gain $\tilde K$ has been implemented into the model of simulations so as to verify the stability of the system. The results have been obtained for a single reach with two underflow gates. The aim is to compare the simulations and experimental curves obtained with this method and the ones obtained experimentally by Dos Santos Martins in this works [8, 10].

4. SIMULATIONS RESULTS

Firstly, let describe the benchmark used for the simulations and the experimentations, described in the second and third subsections respectively.

4.1 Configuration and data of the channel

For this study, the following set of parameters from the practical Valence's channel (Fig.3) is considered where the data are defined such that:

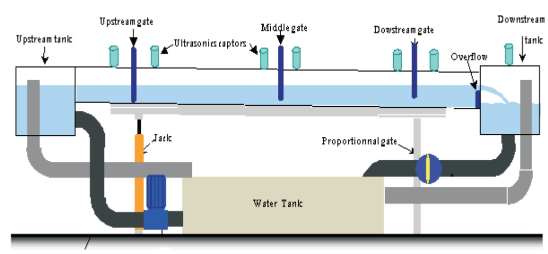

Fig. 3. Valence channel scheme

- $L=64.5dm$ is the length of the channel,
- $b=1dm$ is the width of the channel,
- $N=20$ is the number of the discretizated points,
- Z_L is the water height to regulate, such that $z_{min}<Z<2$ dm; where z_{min} is the minimum critical fluvial water level, 2dm is the canal height.

In this single reach with two gates, the regulation of the water height Z_L at $x=L$, is done by controlling the openings $U_0(t)$ and $U_L(t)$ of the gates at upstream and downstream respectively: it is a multi-variable control (cf Fig. (1)).

The equilibria profiles have been chosen such that the calculated control law from the local models can be efficient over all the operating range of the water height [8]. Let notice that it has been experimentally verified that a local model is valid around $\pm 20\%$ of an equilibrium profile. In order to assign references which are included between $0.6dm$ and $2dm$, the operating points at $x = 0$ are the following:

Table 1. Initial set points for the simulations

Simulations		
$z_{e1}(x=0)$	$z_{e2}(x=0)$	$z_{e3}(x=0)$
$0.625dm$	$0.9375dm$	$1.40625dm$

In this application, the weighting function $\mu_i(\zeta(t))$ is equal to 1 if the output's height is included into the validity domain of the model and 0 in the other case for each operating state. The parameter $\zeta(t)$ exclusively depends on the output which is the only one variable of decision in this precise case.

4.2 Simulations

These results are obtained from an IMB Control and a Multi-Models approach with a LMI gain previously calculated. The figure (4) shows that the output converges to the reference even if this one strongly varies (variations$>$ 100%). The reference tracks a slow dynamic and one can see that the convergence of the output is good.

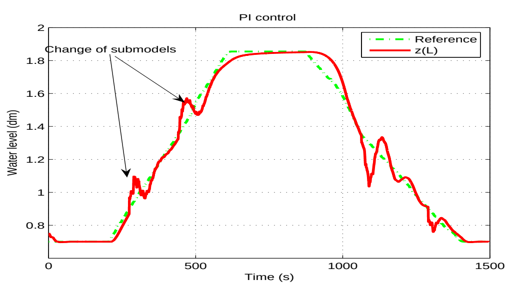

Fig. 4. Variations of the reference along the valued domain

The curves that describe the upstream and downstream gates openings of the reach are given by the figure (5). The convergence of the output to the reference is ensured even when the reference is decreasing or increasing.

The following simulations compare an integral and a PI controllers which gains have been calculated by the LMI approach, Fig (6)-(7). The PI performs better than the integral control as expected.

Next simulations are a comparison between simulations using the theoretical gain obtains through LMI approach (I and PI control) with the first tests realized some years ago by [10], using an experimental Multi-Models gain, without any theoretical study. The figure (8) represents the dynamic evolution of the simulated system and the experimental data obtained with the quite same references.

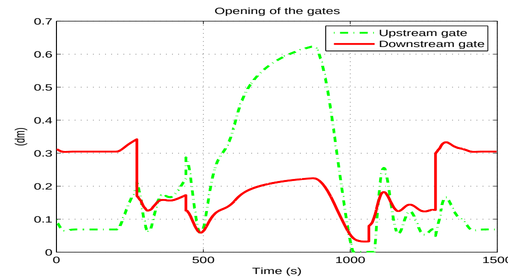

Fig. 5. Gates opening

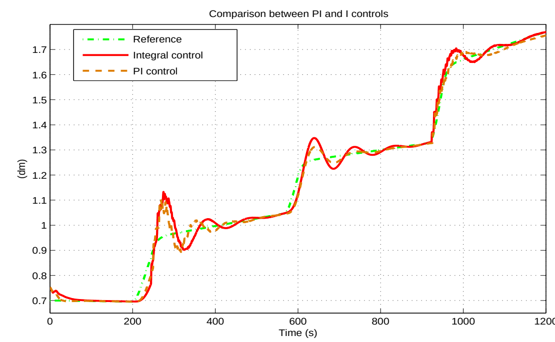

Fig. 6. Comparison of a PI and an integral which gains are simulated with the LMI approach

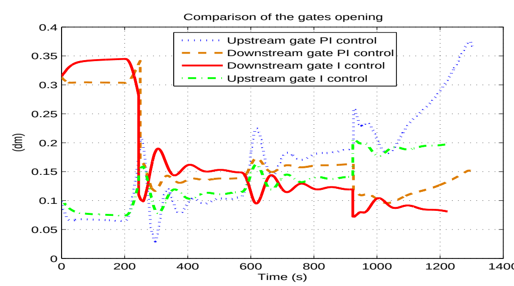

Fig. 7. Gates opening

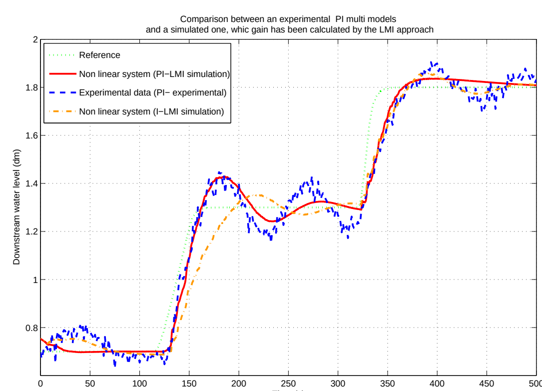

Fig. 8. Comparison of the downstream water level measured in the Valence channel with the first Multi-Models approach in 2004 versus the simulated one with the LMI approach

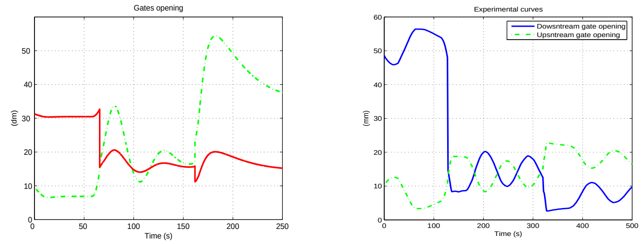

Fig. 9. Gates opening

The figure (9) compares the dynamic of the gate openings.

This study is based on the previous works of Rodrigues [26] and from works of Dos Santos [8, 10].

Experimentations have been realized into the Valence channel (Fig.3) with a Multi-Models approach and a gain calculated via the LMI approach. For these experimentations, the wide range of the accessible water level is attempt for an integral controller, but authors did not get time to implement their PI controller. For the integral case, relevant experimentations have been included into an article, which is actually under revision.

5. CONCLUSION

The first results trough the use of multi-models approach dedicated to irrigation channels control, under an IMBC structure, have been realized some years ago [8]. These good experimental results, but without stabilizable theoretical approach, were obtained. In this paper, the authors have formalized and extended their LMI approach of this regulation problem by the synthesis of a PI Controller.
Simulations have shown the improvements realized towards the initial multi-models approach trough this new PI feedback controller designed by LMI.

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Mobile interface

[CPDE-2013] **DOS SANTOS MARTINS V.**, "Control of a system a coupled PDEs with ODE in infinite dimension ; Application to an extrusion process", **IFAC CPDE 2013**, invited session "PDE Applications" organized by V. Dos Santos Martins, 1st IFAC Workshop on Control of Systems Governed by Partial Differential Equations, September 25-27, 2013, The Institut Henri Poincaré in Paris, France (session invitée)

[CDC-2011] *DIAGNE M.*, **DOS SANTOS MARTINS V.**, COUENNE F., MASCHKE B., "Well posedness of the model of an extruder in infinite dimension", 50th **IEEE Conference on Decision and Control** and European, Control Conference,Orlondo, FL, USA on December 12-15,(n^{o} 1926) 2011

Introduction of a non constant viscosity on an extrusion process: improvements

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Abstract: The aim of this paper is to improve previous works about an extrusion process which is expressed by two systems of conservation laws (with source terms) coupled by a moving interface whose relation is derived from the conservation of momentum. Previous articles have shown the well-posedness and the stability of the linearized system and performed a PI control of the system via an Internal Model Boundary Control structure; for which the parameters have been tuned using the perturbation theory of C_0-semigroup. Those works have been done with simplifying hypothesis like constant viscosity and melt density. Here, the same reasoning is done, taken this time a viscosity and a melt density as functions of the temperature. Some simulations are done with a constant melt density and a viscosity function of the temperature.

Keywords: Coupled systems of conservation laws, IMBControl, infinite dimensional system, Distributed parameter system

1. NOMENCLATURE

B	Geometric parameter
c_p $(J\ kg^{-1}\ K^{-1})$	Specific heat capacity
F $(kg\ s^{-1})$	Mass flow rate
F_d $(kg\ s^{-1})$	Net forward mass flow rate
f $(-)$	Filling ratio
K_d $(-)$	Geometric parameter
l (m)	Moving boundary
M $(-)$	Moisture content
N $(rd\ s^{-1})$	Screw speed
P (Pa)	Pressure
$S_{ech}(m^2)$	Exchange area between melt & barrel
$S_{eff}(m^2)$	Effective area
$T(K)$	Melt temperature
$T_F(K)$	Barrel temperature
$V_{eff}(m^3)$	Effective volume
x (m)	Space coordinate
α $(Jm^{-2}s^{-1}K^{-1})$	Heat exchange coefficient
χ $(-)$	Dimensionless space coord.
$\mu(J\ kg^{-1}\ K^{-1})$	Viscous heat generation parameter
ξ (m)	Pitch length
η $(Pa\ s)$	Melt viscosity
$\rho(kg\ m^{-3})$	Melt density

2. INDEX AND SUPERSCRIPT

in	inlet
p	Partially Filled Zone
f	Fully Filled Zone
e	Equilibrium
$-$	Variable in dimensionless coordinate

3. INTRODUCTION

An extruder is made of a barrel containing one or two Archimedean screws rotating inside the barrel. At the output, the extruder is equipped of a die from which the material is extruded from the process (Fig. (1)). The process is controlled by the barrel temperature and the screw speed to ensure the desired properties (moisture, density, etc... of the food or the polymer) at the die in presence of perturbations. The physical phenomena

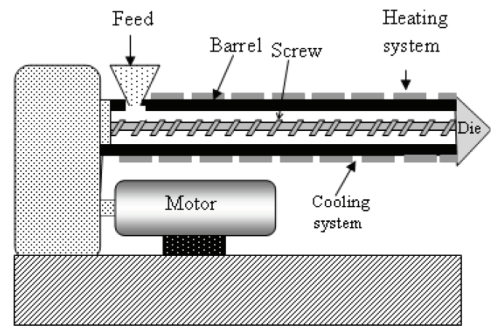

Fig. 1. Description of the mechanism of an extruder

involved in the extrusion process, consist in coupled non linear phenomena, such as viscous Newtonian or non Newtonian fluid flows, heat transfer and possibly chemical reactions. The design of an extrusion process involves a complex modular geometry in function of the screw profile, allowing different capacities of mixing along the extruder. The reader is referred to Vergnes and Berzin (2006) for the steady-state modeling for design purpose and to (Kim and White, 2000a,b; Janssen et al., 2001, 2003; Choulak et al., 2004) for dynamical models and for the control to Kulshreshtha et al. (1995) developing a proportional integral PI feedback or Wang and Tan (2000) developing a multi-variable predictive control.

In previous paper an infinite-dimensional model is developed and analyzed: a simple 1 dimensional model consisting of two systems of conservation laws (with source terms) coupled by a moving interface. In this paper, non constant viscosity and melt density are introduced, and the modifications brought to the model are discussed.

4. THE PHYSICAL MODEL

Following Kulshrestha and Zaror (1992) and Li (2001), the spatial domain of the extruder is split in two parts: the partially and fully filled zones according to the figure (2).

In the partially filled zone (PFZ) (or conveying zone), the pressure is supposed to be constant and equal to the atmospheric pressure P_0. In the fully filled zone FFZ, the filling volume fraction is by definition equal to 1 and the resistance of the die generates a pressure gradient. The difference between the net forward flow at the die and the pumping capacity of the screws causes the displacement of the boundary between the PFZ and FFZ. The dynamic model is derived from the mass and energy balances on a volume element for each zone under some assumptions: the pitch of the screw ξ is uniform; the flow is 1D and strictly convective, the melt density ρ and viscosity η are assumed to be a **function of the temperature** (generally they are inversely proportional to T); there exists a boundary between the PFZ and the FFZ corresponding to discontinuity of the filled volume (or filled volume fraction also called filling ratio); the extruded melt is composed of some species blended with water.

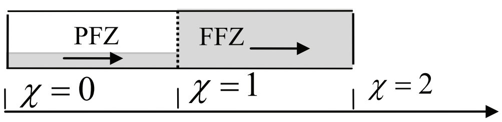

Fig. 2. The 2-zones assumption in the extruder

4.1 Model of the both Zones:

The mass balance equations in the Partially Filled Zone PFZ, are written on the spatial domain $[0, l(t)]$, in terms of the filling ratio f_p (the filled volume fraction which may be related to the total mass density) and the moisture content M_p Kulshrestha and Zaror (1992). The energy balance is written in terms of the of temperature T_p of the mixture.
In the FFZ zone, the model is reduced to the mass balance of water written in terms of the moisture content M_f and the energy balance written in terms of the temperature T_f. The balances are written on the spatial domain $[l(t), L]$.

The balance equations express the convection through the rotation of the screw at the translational velocity, product on the pitch of the screw ξ and the rotation speed of the screw $N(t)$.
The speed of convection $\frac{F_d\xi}{\rho_f V_{eff}}$ is a function of the net flow rate at the die F_d (Eq. (3)), F_d being a function of the geometric characteristic of the die K_d, the viscosity η, the melt density ρ_f and the pressure build-up in this zone $P(x,t)$.

The source term Ω_1 groups the heat produced by the viscosity of the material (proportional to $N^2(t)$) and the heat exchange with the barrel (proportional to $(T_{F_p} - T_p)$). The heat transfer with the barrel and viscous dissipation created by the viscosity are defined in the term Ω_2 for the FFZ.
All the functions depend on (x,t), for an easier lecture they are omitted below [1] :

$$\frac{\partial}{\partial t}\begin{pmatrix} f_p \\ M_p \\ T_p \end{pmatrix} = -\xi N I_3 \frac{\partial}{\partial x}\begin{pmatrix} f_p \\ M_p \\ T_p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_1 \end{pmatrix} \tag{1}$$

$$\text{with } \Omega_1 = \frac{\mu_p \eta N^2(t)}{f_p(x,t)\rho V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho V_{eff} c_p}(T_{F_p} - T_p)$$

$$\text{and } P(x,t) = P_0,\ V_{eff} = \xi S_{eff}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} M_f \\ T_f \end{pmatrix} = \frac{-F_d\xi}{\rho V_{eff}} I_2 \frac{\partial}{\partial x}\begin{pmatrix} M_f \\ T_f \end{pmatrix} + \begin{pmatrix} 0 \\ \Omega_2 \end{pmatrix} \tag{2}$$

$$\text{with } \Omega_2 = \frac{\mu_f \eta N^2(t)}{\rho V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho V_{eff} c_p}(T_{F_f} - T_f)$$

$$\text{and } F_d = \frac{K_d}{\eta}\Delta P,\ \Delta P = (P(L,t) - P_0) \tag{3}$$

The pressure gradient in this zone is expressed as a function of the difference between the maximum flow and F_d:

$$\frac{\partial P(x,t)}{\partial x} = \eta \frac{V_{eff} N(t)\rho - F_d}{B\rho} \tag{4}$$

4.2 Interface relations:

• *Model of the moving interface at $l(t)$:* It is assumed that both zones are separated by an interface defined by the discontinuity of the filling ratio, (Kim and White, 2000a,b): in the PFZ the filling ratio satisfies $f_p(x,t) < 1,\ x \in [0, l(t)[$ with $f_p(l^-,t) < 1$ and in the FFZ $f_p(x,t) = 1,\ x \in]l(t), L]$. The dynamics of the moving boundary is obtained from the global mass balance on the FFZ zone:

$$\frac{dl(t)}{dt} = \frac{F(f_p(l^-,t)) - F_d}{\rho S_{eff}(1 - f_p(l^-,t))} \tag{5}$$

• *At the interface $x = l(t)$*, temperature and moisture content are supposed to be continuous :

$$T_p(l^-,t) = T_f(l^+,t),\ M_p(l^-,t) = M_f(l^+,t)$$

The third coupling relation between the two zones consists in the continuity of the momentum flux (Eq. (6)):

$$\begin{aligned} &F(l^-,t)\xi N(t) + P(l^-,t) f(l^-,t) S_{eff} \\ &= F(l^+,t)\frac{F_d\xi}{\rho V_{eff}} + P(l^+,t) S_{eff} \end{aligned} \tag{6}$$

and allows to compute the mass flow F_d at the die (Eq. (3)) and the transport velocity in the FFZ by integrating the pressure gradient on $[l^+, L]$ (Eq. (4)) and obtaining the pressure:

[1] I_j stands for the identity matrix $j \times j$.

$$P(L,t) = P_0 + \frac{-[1+\frac{K_d}{B}\int_{l^+}^{L}\rho^{-1}dx] + \sqrt{\Delta}}{\frac{2K_d^2}{\eta^2\rho S_{eff}^2}} \quad (7)$$

$$\text{with } \overline{\Omega}_3 = \left(\frac{2K_d}{\eta S_{eff}}\right)^2 \left(\frac{V_{eff}N(t)\int_{l^+}^{L}\eta dx}{B\rho} + \xi^2 N^2(t)\overline{f}_p(1,t) - (1-\overline{f}_p(1,t))\frac{P_0}{\rho}\right) \quad (8)$$

$$\Delta = \left[1+\frac{K_d\int_{l^+}^{L}\rho^{-1}dx]}{B}\right]^2 + \overline{\Omega}_3\left(\overline{f}_p(1,t),N(t),l(t)\right) \quad (9)$$

• *Boundary conditions:* The boundary conditions are defined at the inlet of the extruder that is at $x=0$.The mixing phenomena at the inlet are neglected hence the continuity of the temperature and of the moisture content are assumed, as the fact that the mass flow is continuous and hence equal to the feed rate $F_{in}(t)$ which leads to the boundary condition on the filling ratio :

$$\rho N V_{eff} f_p(0,t) = F_{in}(t) \quad (10)$$
$$T_p(0,t) = T_{in}(t) \quad (11)$$
$$M_p(0,t) = M_{in}(t) \quad (12)$$

where $M_{in}(t)$ and $T_{in}(t)$ are the moisture content and temperature of the matter at the inlet $x=0$.

5. LINEARIZED MODEL EXPRESSED IN FIXED DOMAINS

In order to deal with a system of balance equations in a fixed domain a classical change of spatial variables was performed Diagne et al. (10-2011) for the two zones leading to two systems of conservation laws with source terms and in addition a fictitious convection term due to the change of spatial coordinates.

• *Model in a fixed domain:* For the PFZ (for the FFZ respectively), the change of spatial variables from $[0,l(t)]$ (resp. $x\in(l(t),L)$) onto the interval $(0,1)$ (resp. $(1,2)$) is defined in this way:

$$\chi(x,t) = \frac{x}{l(t)}, \quad \text{respectively } \chi(x,t) = \frac{x+L-2l(t)}{L-l(t)} \quad (13)$$

And the PDE in (1) and in (2) become:

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{f}_p\\ \overline{M}_p\\ \overline{T}_p\end{pmatrix} = \alpha_p I_3 \frac{\partial}{\partial\chi}\begin{pmatrix}\overline{f}_p(\chi,t)\\ \overline{M}_p\\ \overline{T}_p\end{pmatrix} + \begin{pmatrix}0\\0\\ \overline{\Omega_1}\end{pmatrix} \quad (14)$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{M}_f\\ \overline{T}_f\end{pmatrix} = \alpha_f(\chi,t) I_2 \frac{\partial}{\partial\chi}\begin{pmatrix}\overline{M}_f\\ \overline{T}_f\end{pmatrix} + \begin{pmatrix}0\\ \overline{\Omega_2}\end{pmatrix} \quad (15)$$

with $\alpha_p(\chi,t) = -\frac{1}{l(t)}[\xi N(t) - \chi\frac{dl(t)}{dt}]$
and $\alpha_f(\chi,t) = -\frac{1}{L-l(t)}[\frac{F_d\xi}{\rho V_{eff}} - (2-\chi)\,\frac{dl(t)}{dt}]$ and

$$\overline{\Omega_1} = \frac{\mu\eta N^2(t)}{\overline{f}_p(\chi,t)\rho V_{eff}C} + \frac{S_{ech}\alpha}{\rho V_{eff}C}(T_F-\overline{T}_p)$$
$$\overline{\Omega_2} = \frac{\mu\eta N^2(t)}{\rho V_{eff}C} + \frac{S_{ech}\alpha}{\rho V_{eff}C}(T_F-\overline{T}_f)$$

With those news coordinates, the model equations include one fictive convective term depending on the velocity $\frac{dl(t)}{dt}$ of the boundary Diagne et al. (10-2011):

$$\frac{dl(t)}{dt} = \frac{F_d - \rho N(t)V_{eff}\overline{f}_p(1^-,t)}{\rho S_{eff}(1-\overline{f}_p(1^-,t))} \quad (16)$$

• *Equilibrium profiles*: The variables f_p and M_p are constant in time and space as it is shown in this equality:

$$\partial_t\begin{pmatrix}\overline{f}_{pe}\\ \overline{M}_{pe}\\ \overline{M}_{fe}\end{pmatrix} = \partial_\chi\begin{pmatrix}\overline{f}_{pe}\\ \overline{M}_{pe}\\ \overline{M}_{fe}\end{pmatrix} = 0 \quad (17)$$

The temperatures $\overline{T}$ are given by an ODE in χ:

$$\partial_\chi\overline{T}_{pe}(\chi) = \frac{l_e}{\xi N_e}\overline{\Omega}_{1e} \quad (18)$$
$$\partial_\chi\overline{T}_{fe}(\chi) = \frac{(L-l_e)\rho V_{eff}}{\xi F_{de}}\overline{\Omega}_{2e} \quad (19)$$

The moving boundary $l(t)$ is fixed at the equilibrium and induces the following relation between the net flow F_{de} at the die and the screw rotational velocity N_e:

$$l_e = L - \frac{\frac{BP_0}{\rho_e\xi N S_{eff}} + \frac{B\eta_e f_{p,e}}{Kd} - \frac{B\xi N f_{p,e}(1-f_{p,e})}{S_{eff}}}{\frac{1}{\rho_e}\int_1^2\eta d\chi - f_{p,e}\eta_e\int_1^2\frac{1}{\rho}d\chi} \quad (20)$$

• *Definition of the variables:* they are defined in the domain $(0,\,2)$ such that the restrictions on each subdomains are equals to the value of the variable. For the moisture and the temperatures (inside and of the barrel), they are defined on $(0,\,1)\cup(1,\,2)$ by:

$$\delta\overline{M}(\chi,t) = \delta\overline{M}_p(\chi,t)\mathbf{1}_{(0,1)}(\chi) + \delta\overline{M}_f(\chi,t)\mathbf{1}_{(1,2)}(\chi)$$
$$\delta\overline{T}(\chi,t) = \delta\overline{T}_p(\chi,t)\mathbf{1}_{(0,1)}(\chi) + \delta\overline{T}_f(\chi,t)\mathbf{1}_{(1,2)}(\chi)$$
$$\delta\overline{T}_F(\chi,t) = \delta\overline{T}_{F_p}(\chi,t)\mathbf{1}_{(0,1)}(\chi) + \delta\overline{T}_{F_f}(\chi,t)\mathbf{1}_{(1,2)}(\chi)$$

where $\mathbf{1}_{(a,b)}(\chi) = 1$ when $\chi\in(a,b)$, else 0. The same is done for the viscosity and the melt density as they are function of the temperature $\delta\overline{T}$.

The variation of the filling ratio is extended by 0 on the domain $(1,2)$, as it is constant equal to 1 on it (the same is done for the position $\delta l\,(t)$ of the interface):

$$\delta\overline{f}_p(\chi,t) = \delta\overline{f}_p(\chi,t)\mathbf{1}_{(0,1)}(\chi) + 0.\;\mathbf{1}_{(1,2)}(\chi)$$
$$\delta l(\chi,t) = \delta l(t).\mathbf{1}_{(0,1)}(\chi) + 0.\;\mathbf{1}_{(1,2)}(\chi)$$

• *Definition of the state representation*: the evolution of the linearized equations of the PFZ (14), and of the FFZ (15) and of the interface dynamique (16) are given by:

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\overline{f}_p\\ \delta\overline{M}\\ \delta\overline{T}\\ \delta l\end{pmatrix} = A\begin{pmatrix}\delta\overline{f}_p\\ \delta\overline{M}\\ \delta\overline{T}\\ \delta l\end{pmatrix} + B\begin{pmatrix}\delta N\\ \delta T_F\end{pmatrix} + \mathcal{C}\begin{pmatrix}\delta\eta\\ \delta\rho\end{pmatrix} \quad (21)$$

The operators A and B are recalled: A is decomposed as the sum of two operators: $A = A_1 + A_2$. The operator A_1 is defined by:

$$A_1(\chi) = diag\left(\frac{-\xi N_e}{l_e}\mathbf{1}_{(0,1)}(\chi)\partial_\chi, \theta(\chi), \theta(\chi), \alpha_l\right) \quad (22)$$

with $\theta(\chi)$ which deals with the transport equations on each zones:

$$\theta(\chi) = \frac{-\xi N_e}{l_e}\mathbf{1}_{(0,1)}(\chi)\frac{\partial}{\partial\chi} + \frac{-1}{L-l_e}\frac{F_{de}\xi}{\rho V_{eff}}\mathbf{1}_{(1,2)}(\chi)\frac{\partial}{\partial\chi}$$

The operator A_2 is defined by:

$$A_2(\chi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{1,3} & 0 & A_{2,3} & A_{2,4} \\ \alpha_f\delta_{(1^-)}(\chi) & 0 & 0 & 0 \end{pmatrix} \quad (23)$$

with:
$$\begin{aligned} A_{1,3} &= \Big[(\beta_{p1,f} + \beta_{p4}\alpha_f)\mathbf{1}_{(0,1)} \\ &\quad + \Big(\beta_{f3}\gamma_f + \beta_{f5}\alpha_f\Big)\mathbf{1}_{(1,2)}\Big]\delta_{(1^-)} \\ A_{2,3} &= \beta_{p1,T}\mathbf{1}_{(0,1)}(\chi) + \beta_{f1,T}\mathbf{1}_{(1,2)}(\chi) \\ A_{2,4} &= (\beta_{p3} + \beta_{p4}\alpha_l)\,\mathbf{1}_{(0,1)}(\chi) \\ &\quad + (\beta_{f3}\gamma_l + \beta_{f4} + \beta_{f5}\alpha_f)\,\mathbf{1}_{(1,2)}(\chi) \end{aligned}$$

where $\delta_{(1^-)}$ is given by $\delta_{(1^-)}\left(\delta\overline{f}_p(\chi,t)\right) = \delta\overline{f}_p(1^-,t)$.

The input operator is:

$$B(\chi) = \begin{pmatrix} 0 & 0 & \beta_{2,N} & \alpha_N \\ 0 & 0 & \beta_{2,T} & 0 \end{pmatrix}^t \quad (24)$$

$$\begin{aligned} \text{where } \beta_{2,N} &= (\beta_{p2,N} + \beta_{p4}\alpha_N)\mathbf{1}_{(0,1)}(\chi) \\ &\quad + (\beta_{f2,N} + \beta_{f3}\gamma_N + \beta_{f5}\alpha_N)\mathbf{1}_{(1,2)}(\chi) \\ \beta_{2,T} &= \beta_{p2,T}\mathbf{1}_{(0,1)}(\chi) + \beta_{f2,T}\mathbf{1}_{(1,2)}(\chi) \quad (25) \end{aligned}$$

The coefficients are defined by,

$$\text{with } \theta_p = \frac{\mu\eta_e N_e}{\rho_e V_{eff} C} \text{ and } \theta_f = \frac{S_{ech}\alpha}{\rho_e V_{eff} C}$$

$$\beta_{p1,f} = -\frac{\theta_p N_e}{\overline{f}_{pe}^2},\ \beta_{p2,T} = \theta_f = -\beta_{p1,T},\ \beta_{p3} = \frac{\Omega_{1,e}}{l_e},$$

$$\beta_{p2,N} = \frac{\theta_p}{\overline{f}_{pe}} - \frac{\theta_f\alpha(T_{F_{pe}} - \overline{T}_{pe})}{N_e},\ \beta_{p4} = \frac{\chi\Omega_{1,e}}{\xi N_e}$$

$$\text{and} \quad \beta_{f5} = (2-\chi)\frac{\rho_e V_{eff}}{\xi F_{de}}\Omega_{2,e}\ \beta_{f4} = \frac{-\Omega_{2,e}}{(L-l_e)}$$

$$\beta_{f1,T} = -\theta_f = -\beta_{f2,T},\ \beta_{f2,N} = 2\theta_p,\ \beta_{f3} = \frac{\Omega_{2,e}}{\Delta P_e}$$

The novelty is introduced by the third operator, $\mathcal{C}$ expressed independently even if it clearly do not modifies the kernel operator A_1 but the temperature evolution in both parts of the operator A_2:

$$\mathcal{C}(\chi) = \begin{pmatrix} 0 & 0 & \gamma_T & \dfrac{\gamma_l}{\eta} \\ 0 & 0 & \dfrac{\Omega_{1,e}}{\rho_e}\mathbf{1}_{(0,1)}(\chi) & \dfrac{\gamma_l}{\rho} \end{pmatrix}^t \quad (26)$$

with $\gamma_T = \frac{-\beta_{p1,f}}{\eta_e}\mathbf{1}_{(0,1)}(\chi) + \frac{\beta_{f2,N}+\beta_{f2,T}}{\eta_e}\mathbf{1}_{(1,2)}(\chi)$ and $\gamma_l = -\frac{\Omega_{3,e}}{\Delta_e} - \frac{K_d\Delta P}{\eta_e\rho_e S_{eff}(1-\overline{f}_{pe})}$.

The question is now: How this operator $\mathcal{C}$ is going to modify the open loop structure and properties, as the stability, and what are the consequence on the closed loop properties, and the control done previously?

- *Definition of the space state* Dos Santos Martins et al. (07/2012):

$$X = H^1(0,1)\times\mathcal{K}_{(1,2),0}\times\left(H^1(0,2)\right)^2\times\left(\mathcal{K}_{(0,1)}\times\mathcal{K}_{(1,2),0}\right)$$

which can be used with a Hilbert space structure induced by the one of $H^1(0,1)\times\left(H^1(0,2)\right)^2\times\mathbb{R}$ and the norm used is noted by $\|\ \|$.

6. IMPACT OF THE OPERATOR $\mathcal{C}$

For η and ρ constants, the well-posedness and the stability of the open loop have been proved in Diagne et al. (10-2011), Dos Santos Martins et al. (07/2012), using the semi-group theory and the resolvent $R(\lambda, A)$ properties of $A = A_1\partial_x + A_2$. Indeed, $A_1\partial_x$ generates a C_0-semigroup exponentially stable and A_2 is viewed as a bounded perturbation of $A_1\partial_x$ Kato (1976). The same is done with the control part Bu which can be also assimilated to a bounded perturbation, under regularity assumption on u.

6.1 Laws of η and ρ and open loop stability

The laws which described the dynamic of the melt density and the viscosity are numerous and depend on the characteristics of the melt used; food, polymers, etc... Usually, for food, the laws of the evolution of the viscosity (and it is quite the same for the melt density) is a function to the temperature.

Here, it is proposed to define an empirical law about those both variables as follow:

Proposition 1. If it exists two bounded functions y_η and y_ρ such that:

$$\delta\rho = y_\rho\delta\overline{T} = y_{\rho,p}\delta\overline{T}_p\mathbf{1}_{(0,1)}(\chi) + y_{\rho,f}\delta\overline{T}_f\mathbf{1}_{(1,2)}(\chi) \quad (27)$$
$$\delta\eta = y_\eta\delta\overline{T} = y_{\eta,p}\delta\overline{T}_p\mathbf{1}_{(0,1)}(\chi) + y_{\eta,f}\delta\overline{T}_f\mathbf{1}_{(1,2)}(\chi). \quad (28)$$

Then, the matrix A_2 becomes (26):

$$A_2(\chi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{1,3} & 0 & A^*_{2,3} & A_{2,4} \\ \alpha_f\delta_{(1^-)}(\chi) & 0 & \gamma_l\left(\dfrac{y_\eta}{\eta} + \dfrac{y_\rho}{\rho}\right) & 0 \end{pmatrix} \quad (29)$$

with $A^*_{2,3} = A_{2,3} + \gamma_T y_\eta + \frac{\Omega_{1,e}}{\rho_e}\mathbf{1}_{(0,1)}(\chi)y_\rho$.
And the operator A is still stable as A_2 is yet a bounded perturbation of the exponentially stable operator A_1.

6.2 PI control using Internal Model Boundary Control (IMBC): Stability

The control of an extrusion process is complicated, as the variables controllable, the ones measured are not all the time the same and depend on the extruded product, its properties and all those parameters change from one product to another one. One choice could be to control the moisture and the temperature at the end of the fully full zone in order to get the properties desired after the die. As the model has been simplified, only the temperature is taken into account.

The control problem is to find the variations of the screw speed $N(t)$ and the barrel temperature $T_F(\chi, t), \forall \chi \in [0, 2]$ such that temperature at the end of the FFZ, $T_f(L, t)$, tracks a reference signal $r(t)$. The reference signal $r(t)$ is chosen for all cases either constant or non-persistent (a stable step answer of a non-oscillatory system). In this paper, the control scheme based on the Internal Model Boundary Control (IMBC) Dos Santos et al. (2005) is adopted (Figure (3)). This control strategy integrates the process model in real time and allows to regulate the temperature in all the points of the barrel by taking into account the error between the linearized model and the real system (or the non linear model for the simulations).

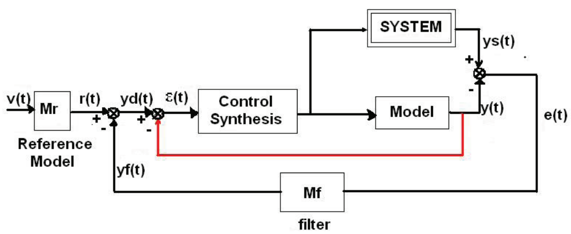

Fig. 3. IMBC structure: Internal Model Boundary Control

• *Closed Loop Stability:* The closed loop system is stable using the same theoretical results as before: the closed loop system is viewed as a perturbation of the open loop one Kato (1976), Pohjolainen (1982). It can be proved, that the closed loop system operator generates a C_0-semigroup under a proportional integral control

$$u(\chi, t) = \alpha_i K_i \int_0^t y(\chi, \tau) - r(\tau) d\tau + \alpha_p K_p [y(\chi, t) - r(t)]$$

with $\alpha_i,\ \alpha_p \in \mathbb{R}^+$ and $K_i,\ K_p \in \mathbb{C}^{p \times p}$, $p = dim(Y),\ y \in Y$. Let

- $\varphi(\chi, t)$ be the open loop state variable,
- $\zeta(\chi, t) = \int_0^t y(\chi, \tau) - r(\tau) d\tau = \int_0^t C\varphi(\chi, \tau) - r(\tau) d\tau$
- ξ is the closed loop state variable and $\xi = \begin{pmatrix} \varphi & \zeta \end{pmatrix}^T$

the closed loop system is, with $A = A_1 \partial_x + A_2$:

$$\dot{\xi} = \begin{pmatrix} A + B\alpha_p K_p C & B\alpha_i K_i \\ C & 0 \end{pmatrix} \xi - \begin{pmatrix} B\alpha_p K_p \\ 1 \end{pmatrix} r(t)$$

$$\dot{\xi}(\chi, t) = \mathcal{A}(\chi)\xi(\chi, t) + \mathcal{B}r(t) \qquad (30)$$

Given to Pohjolainen (1982), appropriate choices for tuning the controller parameters are, under the condition that $rg(CA^{-1}B) = p$ and that A satisfies the spectrum decomposition assumption:

- $0 \leq \alpha_p < \left(\sup_{\lambda \in \Gamma_1} a\|R(\lambda; A)\| + b\|AR(\lambda; A)\|\right)^{-1}$
- $0 \leq \alpha_i < \min_{\lambda \in \Gamma_0} [\|B_e K_i\|^{-1}, \|R(\lambda; A_e)\|^{-1}]$
- $K_p = -[C^+ B^+]^{-1}$ and $K_i = +[CA^{-1}B]^{-1}$

with $P = \frac{-1}{2\pi i} \int_\Gamma R(\lambda; A)\ d\ \lambda$, $PB = \begin{bmatrix} B^+ & 0 \end{bmatrix}^T$, $CP = [C^+\ 0]$, $B_e K_i = \begin{pmatrix} 0 & B\alpha_i K_i \\ 0 & 0 \end{pmatrix}$, $A_e = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.

Those conditions are necessary and sufficient conditions to stabilize the closed loop system and guarantee its robustness. They are satisfied by the the extrusion process described below with η and ρ constant. As only the constant operator A_2 is modified when those both parameters are functions of the temperature, it does not impact on the conditions themselves, only on the values of the control parameters which have to be tuned.

7. SIMULATIONS

The data used are the following:

$L = 2m$	$\rho_0 = 1400kg.m^{-3}$
$B = 2.4 * 10^{-10}$	$Cp = 3.6 * 10^3 Jkg^{-1}\ K^{-1}$
$\eta_0 = 500Pa/s$	$\alpha = 10410Jm^{-2}s^{-1}K^{-1}$
$M_{in} = 0.25$	$F_{in} = 0.025kg/s$
$TFe = 330K$	$\xi = 30 * 10^{-3}m$
$T_{in} = 293K$	$K_d = 0.75 * 10^{-7}$

The viscosity law is given by Khalifeh and Clermont (2007) the WLF (William-Landel-Ferry) law:

$$\eta(T) = \eta(T_0)a_T(T),\ log(a_T) = -\frac{C_1(T - T_0)}{C_2 + T - T_0} \qquad (31)$$

$$\delta\eta = y_{eta}\delta(\bar{T}),\ y_{eta} = -\eta(T_e)\frac{C_1 C_2}{(C_2 + T_e - T_0)^2} \qquad (32)$$

where $\eta(T_0)$ is the viscosity at the initial temperature T_0. C_1 and C_2 are constants depending generally on the material. In a first step, the melt density is supposed constant. The profil equilibrium of the temperature has a physical meaning improved by the introduction of a non constant viscosity (fig. (4))

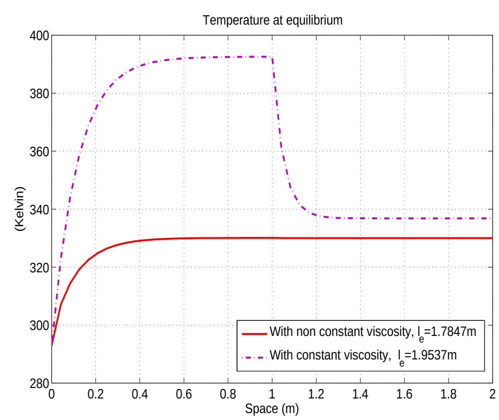

Fig. 4. Temperature profil at equilibrium

The control aim is to regulate the temperature at the end of the barrel, just before the die, such that it tracks a reference signal (5). This figure shows that the temperature tracks correctly the temperature desired in both cases, i.e. with a constant viscosity and a non constant one. The results are quite similar, only the linearized model is better with a non constant viscosity than with a constant one. The remark that can be done is that the internal boundary l_e is also improved at equilibrium by the introduction of a non constant viscosity, cf. fig. (4). The evolution of the temperature inside the barrel and of the control are given in figure (6), in function of the time.

Well posedness of the model of an extruder in infinite dimension

Mamadou Diagne and Valérie Dos Santos Martins and Françoise Couenne and Bernhard Maschke

***Abstract*— The topic of this paper is to present and analyse a physical model of the extrusion process which is expressed two systems of conservation laws (with source terms) coupled by a moving interface whose relation is derived from the conservation of momentum. After a change of variables on the spatial variables is performed in order to transform the time-varying spatial domain in fixed one, the linearisation of the model around an equilibrium profile is given, the well-posedness in the sense of the existence of a C_0-semigroup of those the coupled systems is proven.**

***Index Terms*— Coupled systems of conservation laws, Well-posedness, infinite dimensional system, Distributed parameter system**

NOMENCLATURE

B	Geometric parameter
c_p $(J\ kg^{-1}\ K^{-1})$	Specific heat capacity
F $(kg\ s^{-1})$	Mass flow rate
F_d $(kg\ s^{-1})$	Net forward mass flow rate
f $(-)$	Filling ratio
K_d $(-)$	Geometric parameter
l (m)	Moving boundary
M $(-)$	Moisture content
N $(rd\ s^{-1})$	Screw speed
P (Pa)	Pressure
$S_{ech}(m^2)$	Exchange area between melt & barrel
$S_{eff}(m^2)$	Effective area
$T(K)$	Melt temperature
$T_F(K)$	Barrel temperature
$V_{eff}(m^3)$	Effective volume
x (m)	Space coordinate
α $(Jm^{-2}s^{-1}K^{-1})$	Heat exchange coefficient
χ $(-)$	Dimensionless space coord.
η $(Pa\ s^{-1})$	Melt viscosity
$\mu(J\ kg^{-1}\ K^{-1})$	Viscous heat generation parameter
$\rho_0(kg\ m^{-3})$	Melt density
ξ (m)	Pitch lenght

INDEX AND SUPERSCRIPT

in	inlet
p	Partially Filled Zone
f	Fully Filled Zone
e	Equilibrium
—	Variable in dimensionless coordinate

I. INTRODUCTION

An extruder is made of a barrel containing one or two Archimedean screws rotating inside the barrel. At the output, the extruder is equipped of a die from which the material is extruded from the process (Fig. 1). The process is controlled by the barrel temperature and the screw speed to ensure the desired properties (moisture, density, etc... of the food or the polymer) at the die in presence of perturbations. The physical phenomena involved in the extrusion process,

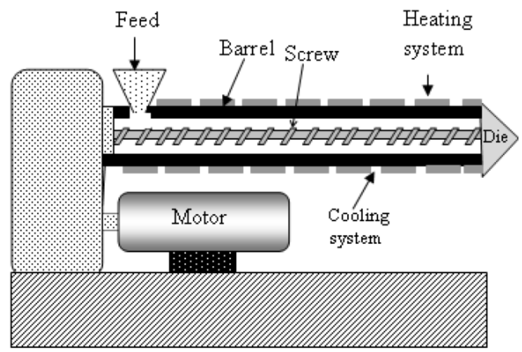

Fig. 1. Description of the mechanism of an extruder

consist in coupled non linear phenomena, such as viscous Newtonian or non Newtonian fluid flows, heat transfer and possibly chemical reactions. The design of an extrusion process involves a complex modular geometry in function of the screw profile, allowing different capacities of mixing along the extruder. The reader is referred to [1] for the steady-state modelling for design purpose and to [2], [3], [4], [5], [6] for dynamical models and for the control to [7] developing a proportional integral PI feedback or [8] developing a multi-variable predictive control.

But the main subject that we shall be interested in, is that the extruder is divided in time-varying spatial zones where the material fills completely or not the volume of the extruder involving a change of causality and order of the system. In this paper we shall follow [9], [10] where an infinite-dimensional model is developed and shall define and analyse a simple 1 dimensional model consisting of two systems of conservation laws (with source terms) coupled by a moving interface.

In the second section the physical model of the extrusion process is recalled in terms of two systems of conservation laws (with source terms) coupled by a moving interface whose relation is derived from the conservation of momentum. In the third section a change of variables on the spatial variables is performed in order to transform the time-varying spatial domain in fixed one, thereby introducing a fictitious convection term in the conservation laws. In the fourth section this model is linearized. In the fifth section, the dynamics of the boundary is integrated to the distributed

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state variables and the obtained linear system is shown to generate a C_0-semigroup.

II. The physical model

Following [9] and [10], the spatial domain of the extruder is split in two parts: the partially and fully filled zones according to the figure 2.

In the partially filled zone (PFZ) (or conveying zone), the pressure is supposed to be constant and equal to the atmospheric pressure P_0. In the fully filled zone FFZ, the filling volume fraction is by definition equal to 1 and the resistance of the die generates a pressure gradient. The difference between the net forward flow at the die and the pumping capacity of the screws causes the displacement of the boundary between the PFZ and FFZ. The dynamic model is derived from the mass and energy balances on a volume element for each zone under some assumptions:

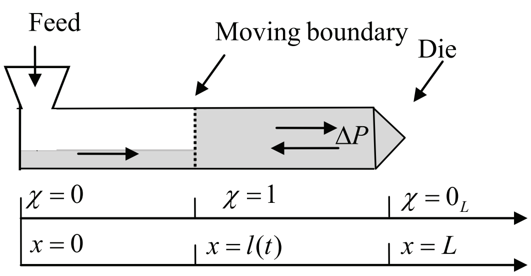

Fig. 2. The 2-zones assumption in the extruder

- the pitch of the screw ξ is uniform;
- the flow is 1D and strictly convective, the melt density ρ_0 and viscosity η are assumed to be constant;
- there exists a boundary between the PFZ and the FFZ corresponding to discontinuity of the filled volume (or filled volume fraction also called filling ratio);
- the extruded melt is composed of some species blended with water

A. Model of the Partially Filled Zone (PFZ):

The mass balance equations in the PFZ, are written on the spatial domain $[0, l(t)]$, in terms of the filling ratio f_p (the filled volume fraction which may be related to the total mass density) and the moisture content M_p [9]. The energy balance is written in terms of the of temperature T_p of the mixture.

The balance equations express the convection through the rotation of the screw at the translational velocity, product on the pitch of the screw ξ and the rotation speed of the screw $N(t)$. The source term Ω_1 groups the heat produced by the viscosity of the material (proportional to $N^2(t)$) and the heat exchange with the barrel (proportional to $(T_{F_p} - T_p)$)[1]:

$$\frac{\partial}{\partial t}\begin{pmatrix} f_p(x,t) \\ M_p(x,t) \\ T_p(x,t) \end{pmatrix} = -\xi N(t) I_3 \frac{\partial}{\partial x}\begin{pmatrix} f_p(x,t) \\ M_p(x,t) \\ T_p(x,t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_1(f_p, N(t), T_p, T_{F_p}) \end{pmatrix} \tag{1}$$

$$\begin{aligned} \text{with} \quad & \Omega_1 = \frac{\mu_p \eta_p N^2(t)}{f_p(x,t)\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_p} - T_p) \\ \text{and} \quad & P(x,t) = P_0. \\ & V_{eff} = \xi S_{eff} \end{aligned}$$

B. Model of the Fully Filled Zone FFZ:

In the FFZ zone, the model is reduced to the mass balance of water written in terms of the moisture content M_f and the energy balance written in terms of the temperature T_f. The balances are written on the spatial domain $[l(t), L]$. The speed of convection $\frac{F_d \xi}{\rho_0 V_{eff}}$ is a function of the net flow rate at the die F_d (Eq. 3), F_d being a function of the geometric characteristic of the die K_d, the viscosity η_f and the pressure build-up in this zone $P(x,t)$. The heat transfer with the barrel and viscous dissipation created by the viscosity are defined in the term Ω_2.

$$\frac{\partial}{\partial t}\begin{pmatrix} M_f(x,t) \\ T_f(x,t) \end{pmatrix} = \frac{-F_d \xi}{\rho_0 V_{eff}} I_2 \frac{\partial}{\partial x}\begin{pmatrix} M_f(x,t) \\ T_f(x,t) \end{pmatrix} + \begin{pmatrix} 0 \\ \Omega_2(N(t), T_f, T_{F_f}) \end{pmatrix} \tag{2}$$

$$\text{with } \Omega_2 = \frac{\mu_f \eta_f N^2(t)}{\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_f} - T_f)$$

$$\text{and } F_d = \frac{K_d}{\eta_f}\Delta P \tag{3}$$

$$\Delta P = (P(L,t) - P_0) \tag{4}$$

The pressure gradient in this zone is expressed as a function of the difference between the maximum flow and F_d:

$$\frac{\partial P(x,t)}{\partial x} = \eta_f \frac{V_{eff} N(t) \rho_0 - F_d}{B \rho_0} \tag{5}$$

Let us note that, as a consequence of a constant melt density, the gradient (5) is uniform.

C. Model of the moving interface at $l(t)$:

Following [2], [3], we assume that the two zones are separated by an interface defined by the discontinuity of the filling ratio: in the PFZ the filling ratio satisfies $f_p(x,t) < 1, x \in [0, l(t)[$ with $f_p(l^-, t) < 1$ and in the FFZ $f_p(x,t) = 1$, $x \in]l(t), L]$. The dynamics of the moving boundary is obtained from the global mass balance on the FFZ zone:

$$\frac{dl(t)}{dt} = \frac{F(f_p(l^-, t)) - F_d}{\rho_0 S_{eff}(1 - f_p(l^-, t))} \tag{6}$$

[1] I_j stands for the identity matrix $j \times j$.

D. Interface relations:

At the interface $x = l(t)$, temperature and moisture content are supposed to be continuous :

$$\begin{aligned} T_p(l^-,t) &= T_f(l^+,t) \\ M_p(l^-,t) &= M_f(l^+,t) \end{aligned}$$

The third coupling relation between the two zones consists in the continuity of the momentum flux (Eq. 7):

$$\begin{aligned} & F(l^-,t)\xi N(t) + P(l^-,t) f(l^-,t) S_{eff} \\ = \;& F(l^+,t)\frac{F_d \xi}{\rho_0 V_{eff}} + P(l^+,t) S_{eff} \end{aligned} \tag{7}$$

and allows to compute the mass flow F_d at the die (eq. (3)) and the transport velocity in the FFZ by integrating the pressure gradient on $[l^+, L]$ (Eq. 5) and obtaining the pressure:

$$P(L,t) - P_0 = \frac{-[1 + \frac{K_d}{B\rho_0}(L - l^+)] + \sqrt{\Delta}}{\frac{2K_d^2}{\eta_f^2 \rho_0 S_{eff}^2}} \tag{8}$$

with $\Delta = [1 + \frac{K_d}{B\rho_0}(L - l^+)]^2 + \Omega_3\left(f_p(l^-,t), N(t), l^+\right)$

$$\begin{aligned} \text{and } \Omega_3 \;=\; & \left(\frac{2K_d}{\eta_f S_{eff}}\right)^2 \left(\frac{\eta_f V_{eff} N(t)}{B\rho_0}(L - l^+)\right. \\ & \left. + \xi^2 N^2(t) f_p(l^-,t) - (1 - f_p(l^-,t))\frac{P_0}{\rho_0}\right) \end{aligned}$$

E. Boundary conditions:

The boundary conditions are defined at the inlet of the extruder that is at $x = 0$. It is assumed that the mass flow is continuous and hence equal to the feed rate $F_{in}(t)$ which leads to the boundary condition on the filling ratio :

$$f_p(0,t) = \frac{F_{in}(t)}{\rho_0 N V_{eff}} \tag{9}$$

The mixing phenomena at the inlet are neglected hence the continuity of the temperature and of the moisture content are assumed :

$$\begin{aligned} T_p(0,t) &= T_{in}(t) \\ M_p(0,t) &= M_{in}(t) \end{aligned}$$

where $M_{in}(t)$ and $T_{in}(t)$ are the moisture content and temperature of the matter at the inlet $x = 0$.

III. Model expressed in fixed domains

In order to deal with a system of balance equations in a fixed domain a classical change of spatial variables is performed for the two zones leading to two systems of conservation laws with source terms and in addition a fictitious convection term due to the change of spatial coordinates.

A. Partially filled zone with fixed boundary model

The change of spatial variables from $[0, l(t)]$ onto the interval $[0, 1]$ is defined in this way:

$$\chi(x,t) = \frac{x}{l(t)} \tag{10}$$

And the PDE in (1) becomes:

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} \overline{f}_p(\chi,t) \\ \overline{M}_p(\chi,t) \\ \overline{T}_p(\chi,t) \end{pmatrix} &= \alpha_p(\chi,t) I_3 \frac{\partial}{\partial \chi}\begin{pmatrix} \overline{f}_p(\chi,t) \\ \overline{M}_p(\chi,t) \\ \overline{T}_p(\chi,t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ \overline{\Omega}_1(\overline{f}_p, N(t), \overline{T}_p, T_{F_p}) \end{pmatrix}, \chi \in (0,1) \end{aligned} \tag{11}$$

with $\alpha_p(\chi,t) = -\frac{1}{l(t)}[\xi N(t) - \chi \frac{dl(t)}{dt}]$

and $\overline{\Omega}_1 = \frac{\mu_p \eta_p N^2(t)}{\overline{f}_p(\chi,t)\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_p} - \overline{T}_p)$

With those news coordinates, the model equations include one fictive convective term depending on the velocity $\frac{dl(t)}{dt}$ of the boundary .

B. Fully filled zone with fixed boundary model

In this zone, the change of spatial variables from $x \in (l(t), L)$ onto the interval $[0, 1]$ is defined by :

$$\chi(x,t) = \frac{L - x}{L - l(t)} \tag{12}$$

And the PDE in (2) becomes:

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} \overline{M}_f(\chi,t) \\ \overline{T}_f(\chi,t) \end{pmatrix} &= \alpha_f(\chi,t) I_2 \frac{\partial}{\partial \chi}\begin{pmatrix} \overline{M}_f(\chi,t) \\ \overline{T}_f(\chi,t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \overline{\Omega}_2(N(t), \overline{T}_f, T_{F_f}) \end{pmatrix}, \;\; \chi \in (0,1) \end{aligned} \tag{13}$$

with $\alpha_f(\chi,t) = -\frac{1}{L - l(t)}[\frac{F_d \xi}{\rho_0 V_{eff}} + \chi \frac{dl(t)}{dt}]$

with $\overline{\Omega}_2 = \frac{\mu_f C \eta_f N^2(t)}{\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_f} - \overline{T}_f)$

The net flow at the die F_d is given by those expressions:

$$F_d = \frac{K_d}{\eta_f}\Delta\overline{P} \tag{14}$$

$$\text{with} \qquad \Delta\overline{P} = (\overline{P}(0,t) - P_0) \tag{15}$$

The boundary conditions and the interface relations are easily deduced from their expression in the original spatial variables and are not developed further.

IV. The linearized model in the fixed boundary coordinates

In this section, the linearization of the system around some equilibrium profile is derived.

A. Equilibrium profiles

- The variables f_p and M_p are constant in time and space as it is shown in this equality:

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{f}_{pe}\\ \overline{M}_{pe}\end{pmatrix} = \frac{\partial}{\partial \chi}\begin{pmatrix}\overline{f}_{pe}\\ \overline{M}_{pe}\end{pmatrix} = 0 \qquad (16)$$

- The temperature $\overline{T}_p$ is given by an ODE in χ:

$$\frac{\partial \overline{T}_{pe}}{\partial \chi}(\chi) = \frac{l_e}{\xi N_e}\overline{\Omega}_{1e} \qquad (17)$$

- The variable $\overline{M}_f$ which describes the moisture concentration is constant in time and space:

$$\frac{\partial}{\partial t}\overline{M}_{fe} = \frac{\partial}{\partial \chi}\overline{M}_{fe} = 0 \qquad (18)$$

- The evolution of the temperature in this zone is driven by a differential equation in χ as in the PFZ:

$$\frac{\partial}{\partial \chi}\overline{T}_{fe}(\chi) = \frac{(L-l_e)\rho_0 V_{eff}}{\xi F_{de}}\overline{\Omega}_{2e} \qquad (19)$$

The moving boundary $l(t)$ is fixed at the equilibrium and induces the following relation between the net flow F_{de} at the die and the screw rotational velocity N_e:

$$\frac{dl_e}{dt} = 0 \Leftrightarrow F_{de} = \rho_0 N_e V_{eff} f_e \qquad (20)$$

B. Linear model around the equilibrium profile

The linearization of the two systems of PDE's in fixed domain and the dynamics of the moving interface is then obtained as follows.
- The PFZ Linearized model is given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\overline{f}_p\\ \delta\overline{M}_p\\ \delta\overline{T}_p\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\\ \beta_{p2,N} & \beta_{p2,T}\end{pmatrix}\begin{pmatrix}\delta N\\ \delta T_{F_p}\end{pmatrix} \qquad (21)$$
$$+\begin{pmatrix}0\\0\\ \beta_{p3}\end{pmatrix}\delta l + \begin{pmatrix}0\\0\\ \beta_{p4}\end{pmatrix}\delta\frac{dl}{dt}$$
$$+\left(-\frac{1}{l_e}\xi N_e I_3 \partial_\chi + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \beta_{p1,f} & 0 & \beta_{p1,T}\end{pmatrix}\right)\begin{pmatrix}\delta\overline{f}_p\\ \delta\overline{M}_p\\ \delta\overline{T}_p\end{pmatrix}$$

with $\beta_{p1,f} = -\dfrac{\mu_p C \eta_p N_e^2}{\overline{f}_{pe}^2 \rho_0 V_{eff} c_p},\ \beta_{p1,T} - \dfrac{S_{ech}\alpha}{\rho_0 V_{eff} c_p},$

$$\beta_{p2,N} = \frac{\mu_p \eta_p N_e}{\overline{f}_{pe}\rho_0 V_{eff} c_p} - \frac{S_{ech}\alpha(T_{F_{pe}} - \overline{T}_{pe})}{N_e \rho_0 V_{eff} c_p}$$
$$\beta_{p2,T} = \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p},\ \beta_{p4} = \chi\frac{l_e}{\xi N_e}\beta_{p3}$$
$$\beta_{p3} = \frac{1}{l_e}\left(\frac{\mu_p \eta_p N_e^2(t)}{\overline{f}_{pe}\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_{pe}} - \overline{T}_{pe})\right)$$

The FFZ Linearized model is given by:

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\overline{M}_f\\ \delta\overline{T}_f\end{pmatrix} = -\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}} I_2 \frac{\partial}{\partial \chi}\begin{pmatrix}\delta\overline{M}_f\\ \delta\overline{T}_f\end{pmatrix} \qquad (22)$$
$$+\begin{pmatrix}0 & 0\\ 0 & \beta_{f1}\end{pmatrix}\begin{pmatrix}\delta\overline{M}_f\\ \delta\overline{T}_f\end{pmatrix} + \begin{pmatrix}0 & 0\\ \beta_{f2,N} & \beta_{f2,T}\end{pmatrix}\begin{pmatrix}\delta N\\ \delta T_{F_f}\end{pmatrix}$$
$$+\begin{pmatrix}0\\ \beta_{f3}\end{pmatrix}\delta\overline{P}(0,t) + \begin{pmatrix}0\\ \beta_{f4}\end{pmatrix}\delta l(t) + \begin{pmatrix}0\\ \beta_{f5}\end{pmatrix}\delta\frac{dl}{dt}$$

with $\beta_{f1} = -\dfrac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}$

$$\beta_{f2,N} = \frac{2\mu_f \eta_f N_e}{\rho_0 V_{eff} c_p},\ \beta_{f2,T} = \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}$$
$$\beta_{f3} = \frac{-K_d\left(\mu_f \eta_f N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})\right)}{\rho_0 \eta_f F_{de} V_{eff} c_p}$$
$$\beta_{f4} = \frac{\mu_f \eta_f N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})}{(L-l_e)c_p}$$
$$\beta_{f5} = -\chi\frac{\mu_f \eta_f N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})}{\xi F_{de} c_p}$$

Linearized dynamics of the moving interface:

$$\frac{d(\delta l)}{dt} = \frac{-K_d}{\rho_0 \eta_f S_{eff}(1-\overline{f}_{pe})}\delta P(0,t) + \frac{\xi\overline{f}_{pe}}{(1-\overline{f}_{pe})}\delta N$$
$$+\left[\frac{\xi N_e(1-2\overline{f}_{pe})}{(1-\overline{f}_{pe})^2} + \frac{K_d(\overline{P}_e - P_0)}{\eta_f \rho_0 S_{eff}(1-\overline{f}_{pe})^2}\right]\delta\overline{f}_p(1^-,t) \qquad (23)$$

Boundary conditions become:

$$\delta\overline{f}_p(0,t) = \delta\frac{F_{in}(t)}{\rho_0 N V_{eff}}$$
$$\delta\overline{T}_p(0,t) = \delta T_{in}(t),\ \delta\overline{M}_p(0,t) = \delta M_{in}(t)$$

and

$$\delta\overline{P}(0,t) = \frac{\delta N}{\sqrt{\overline{\Delta}_e}}\left(\frac{\eta_f V_{eff}}{B}(0-l_e) + 2\rho_0\xi^2 N_e \overline{f}_{pe}\right)$$
$$+\frac{\delta\overline{f}_p}{\sqrt{\overline{\Delta}_e}}(\rho_0\xi^2 N_e^2 + P_0) + \delta l\frac{(\eta_f S_{eff})}{B\sqrt{\overline{\Delta}_e}}\Big[-\xi N_e$$
$$+\frac{(\eta_f S_{eff})}{2K_d}\left(\sqrt{\overline{\Delta}_e} - (1 + \frac{K_d}{B\rho_0}(0-l_e))\right)\Big] \qquad (24)$$

Interface relations are expressed at $\chi = 1$. The continuity of the moisture concentration and the temperature is assumed:

$$\delta\overline{T}_p(1,t) = \delta\overline{T}_f(1,t)$$
$$\delta\overline{M}_p(1,t) = \delta\overline{M}_f(1,t)$$

C. Well posedness of the linearized PFZ & FFZ

The systems associated with each of the zones define control systems. Indeed the operators $\left(-\frac{1}{l_e}\xi N_e I_3 \frac{\partial}{\partial\chi} + \tilde{\beta_{p1}}\right)$ and $\left(-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}} I_2 \frac{\partial}{\partial\chi} + \tilde{\beta_{f1}}\right)$ generate each one a $\mathcal{C}_0$-semigroup as it may be proved using the perturbation theory of operators [11] and results developed for hyperbolic systems [12] for the homogeneous systems (21-22), i.e. $U = (\delta N\ \delta T_F)^T = 0$. It may be easily checked that these operators are closed operators and densely defined in $L_2(0,1)$ (and resp. $L_2(0_L,1)$, $\tilde{\beta}$ stands for the matrix associated to β). Indeed:

- $-\frac{1}{l_e}\xi N_e I_3$ and $\tilde{\beta_{p1}}$ are linear and bounded operators, (resp. $-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2$ and $\tilde{\beta_{f1}}$)
- the domain of $\tilde{\beta_{p1}}$ is dense in $L_2(0,1)$ (resp. $\tilde{\beta_{f1}}$ in $L_2(0_L,1)$),
- $-\frac{1}{l_e}\xi N_e I_3$ is invertible, (resp. $-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2$).

So as the operator ∂_χ generates a $\mathcal{C}_0$-semigroup, then $\left(-\frac{1}{l_e}\xi N_e I_3\frac{\partial}{\partial\chi}+\tilde{\beta_{p1}}\right)$ (resp. $\left(-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2\frac{\partial}{\partial\chi}+\tilde{\beta_{f1}}\right)$) can be viewed as a bounded perturbation (additive and multiplicative one) of the operator ∂_χ. So they still generate a $\mathcal{C}_0$-semigroup [11], [12]. Furthermore the input maps are linear and bounded, hence the systems (21) and (22) define control systems for which the solutions are well-defined.

D. The moving boundary $l(t)$

The linearized dynamics of the moving boundary is defined by replacing (24) into (23) and one obtains the control system:

$$\frac{d(\delta l)}{dt}=\alpha_l\delta l+\alpha_f\delta f_p+\alpha_N\delta N \tag{25}$$

The physically admissible numerical values lead to the positivity of the coefficient α_l hence to an unstable drift system. Such instability is not observed physically and as a conclusion the coupling of the models of the two zones though the interface relations is essential for the well-posedness of the complete system. This will be the topic of the next section.

V. Analysis of the linearized system of the complete 2-zones model

The proof of the existence of solutions for systems of conservation laws through some moving boundary may be analyzed in different ways for instance by considering two systems of PDE's coupled by an ODE and closing the loop after having proved the existence of solutions for the cascaded system [13]. Another approach is to consider a color function, defining the two spatial domain, and augment the state space with this function [14]. In this paper we shall follow some similar route and define a distributed variable associated with the position of the boundary:

$$\delta l(x,t)=\delta l(t).\mathbb{1}_{(0,1)}(\chi) \tag{26}$$

and belongs to the subspace of constant functions $K(0,1)$ (which is isomorphic to $\mathbb{R}$). We shall consider the complete linearized system defined by the state variables $\varphi(\chi,t)$:

$$\varphi^T=\begin{pmatrix}\delta\overline{f}_p & \delta\overline{M}_p & \delta\overline{M}_f & \delta\overline{T}_p & \delta\overline{T}_f & \delta l\end{pmatrix} \tag{27}$$

$\delta\overline{M}_p$, $\delta\overline{M}_f$, $\delta\overline{T}_p$, $\delta\overline{T}_f$ and $\delta\overline{f}_p$ are defined in $H^1(0,1)$, belonging to the state space:

$$X=H^1(0,1)^5\times K(0,1) \tag{28}$$

which is isomorphic to $(H^1(0,1))^5\times\mathbb{R}$. The homogeneous system expression is given then by:

$$\partial_t\varphi(x,t)=A(\chi)\varphi(x,t)=\big(A_1(\chi)+A_2(\chi)\big)\varphi(x,t) \tag{29}$$

The operator $A(\chi)$ can be decomposed in two operators, $A_2(\chi)$ a bounded operator, and $A_1: D(A_1)\subset X\to Y=L^2(0,1)^5\times\mathbb{R}$ composed of the differential operator ∂_χ (for more details, see [15]). Those operators are as follow:

$$A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\A_{2,1}&0&0&A_{2,4}&0&A_{2,6}^p\\A_{2,1}&0&0&0&A_{2,5}&A_{2,6}^f\\0&0&0&0&0&0\end{pmatrix} \tag{30}$$

$$\begin{aligned}\text{with } A_{2,1} &= -\frac{\mu_p C\eta_p N_e^2}{\overline{f}_{pe}^2\rho_0 V_{eff}c_p},\quad A_{2,6}^p=\beta_{p3} && (31)\\ A_{2,4}=A_{2,5} &= -\frac{S_{ech}\alpha}{\rho_0 V_{eff}c_p} && (32)\\ A_{2,6}^f &= \left(\beta_{f3}(\gamma_l-\gamma_f\frac{\alpha_l}{\alpha_f})+\beta_{f4}\right) && (33)\end{aligned}$$

The expression of the differential operator A_1 is:

$$A_1=\begin{pmatrix}\theta^p\partial_\chi&0&0&0&0&0\\0&\theta^p\partial_\chi&0&0&0&0\\0&0&\theta^f\partial_\chi&0&0&0\\\beta_{41}&0&0&\theta^p\partial_\chi&0&\beta_{46}\\\beta_{51}&0&0&0&\theta^f\partial_\chi&\beta_{56}\\\alpha_f\delta_1&0&0&0&0&\alpha_l\end{pmatrix}$$

$$\begin{aligned}\text{with } \theta^p &= -\frac{\xi N_e}{l_e},\theta^f=-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}} && (34)\\ \beta_{41}(\chi) &= \beta_{p4}(\chi)\alpha_f\delta_1(\chi) && (35)\\ \beta_{46}(\chi) &= \beta_{p4}(\chi)\alpha_l && (36)\\ \beta_{51}(\chi) &= \Big(\beta_{f3}(\chi)\gamma_f+\beta_{f5}(\chi)\alpha_f\Big)\delta_1(\chi) && (37)\\ \beta_{56}(\chi) &= \Big(\beta_{f3}(\chi)\gamma_f\frac{\alpha_l}{\alpha_f}+\beta_{f5}(\chi)\alpha_l\Big) && (38)\end{aligned}$$

Corollary 1 ([16], Hille-Yosida): Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of $\mathcal{C}_0$-semigroup satisfying $\|T(t)\|\le e^{wt}$, $w\in\mathbb{R},\forall z\in D(A)$, are:

$$Re(\langle Az,z\rangle)\le w\|z\|^2 \quad\text{for}\quad z\in D(A) \tag{39}$$

$$Re(\langle A^*z,z\rangle)\le w\|z\|^2 \quad\text{for}\quad z\in D(A^*) \tag{40}$$

The operators A_1 satisfies the condition $\forall z\in D(A_1)$ (resp. for $D(A_1^*)$):

$$\langle A_1z,z\rangle\le C\|z\|_{H^1}^2\ ,\langle A_1^*z,z\rangle\le C\|z\|_{H^1}^2 \tag{41}$$

using Holder and the triangular inequalities. Indeed, one gets:

$$\begin{aligned}\langle A_1z,z\rangle &= \int_0^1(A_1z)^Tz && (42)\\ &= \int_0^1\theta^p\partial_\chi z_1z_1+\theta^p\partial_\chi z_2z_2+\theta^f\partial_\chi z_3z_3dx\\ &+ \int_0^1\theta^p\partial_\chi z_4z_4+\theta^f\partial_\chi z_5z_5+\alpha_l\partial_\chi z_6z_6dx\\ &+ \int_0^1\beta_{41}z_1z_4+\beta_{46}z_6z_4+\beta_{51}z_1z_5dx\\ &+ \int_0^1\beta_{56}z_6z_5+\alpha_f\delta_1z_1z_6dx && (43)\end{aligned}$$

Each terms $\int_0^1 \theta\partial_\chi z_i z_i$ can be bounded by:

$$\int_0^1 \theta\partial_\chi z_i z_i dx \leq |\theta| \|z_i\|^2_{H^1(0,1)} \tag{44}$$

In the same way, each coupled product, like $\int_0^1 \beta_{41} z_1 z_4 dx$, can be bounded using Holder inequalities, e.g.:

$$\begin{aligned}\int_0^1 \beta_{41} z_1 z_4 dx &\leq \sup_{(0,1)} |\beta_{41}| \int_0^1 z_1 z_4 dx \\ &\leq C_{41} \left(\|z_1\|^2_{H^1(0,1)} + \|z_4\|^2_{H^1(0,1)} \right) \end{aligned} \tag{45}$$

The same is done for $\int_0^1 \alpha_f \delta_1 z_1 z_6 dx$ recalling that

$$\delta_1 z_1 = z_1(1) = \int_0^1 z_1' dx$$

and that $\|z_1\|^2_{H^1(0,1)} = \|z_1\|^2_{L_2(0,1)} + \|z_1'\|^2_{L_2(0,1)}$. So there exists a constant $C = \sup(|\theta^p|, |\theta^f|, C_{ij})$ such that

$$\langle A_1 z, z\rangle \leq C \left(\sum_1^6 \|z_i\|^2_{H^1(0,1)} \right) = C\|z\|^2_{H^1(0,1)} \tag{46}$$

The same is done with the adjoint operator $A_1^* : Y^* \to X^*$:

$$A_1^* = \begin{pmatrix} -\theta^p\partial_\chi & 0 & 0 & \beta_{41} & \beta_{51} & \alpha_f\delta_1 \\ 0 & -\theta^p\partial_\chi & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta^f\partial_\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta^p\partial_\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta^f\partial_\chi & 0 \\ 0 & 0 & 0 & \beta_{46} & \beta_{56} & \alpha_l \end{pmatrix} \tag{47}$$

and one gets the same constant to bound $\langle A_1^* z, z\rangle$:

$$\langle A_1^* z, z\rangle \leq C \left(\sum_1^6 \|z_i\|^2_{H^1(0,1)} \right) = C\|z\|^2_{H^1(0,1)} \tag{48}$$

and A_1 is the infinitesimal generator of a $\mathcal{C}_0$-semigroup.
All the more, the bounded operator A_2 is a bounded additive perturbation of the operator A_1:

Theorem 1 ([17]): Let X a Banach space and let A the infinitesimal generator of a C_0-semigroup $T(t)$ on X such that $\|T(t)\| \leq Me^{wt}$. If B is a bounded linear operator on X then $A + B$ is infinitesimal generator of a C_0-semigroup $S(t)$ on X such that $\|S(t)\| \leq Me^{(w+M\|B\|)t}$. □

So $A = A_1 + A_2$ still generates a $\mathcal{C}_0$-semigroup $T(t)$ which satisfies $\|T(t)\| \leq e^{(w+\|A_2\|)t}$. The system (29) is well posed [11], [12].
Still using the same results, the system with the control $U(t)$ still generates a $\mathcal{C}_0$-semigroup if bounded inputs are considered and can be viewed as bounded perturbations.

VI. Conclusion

In this paper, a model of an extruder is proposed, which takes into account the moving boundary between the partially and the fully filled zone. The complexity of this system of coupled PDEs and ODE comes from the mobility of the internal interface $l(t)$. A change of space coordinate to define fixed spatial coordinates is developed, and the linearized system is written in those new coordinates. The well posedness of those equations is proved for the coupled systems in the homogeneous case. The system with the control $U(t)$ still generates a $\mathcal{C}_0$-semigroup considering that the variations $(\delta N\ \delta T_F)$ are bounded ones.
The stability problem can then be discussed, noting that if the w of the corollary 1 is negative, then the system is exponentially stable. The majorations realized for the well posedness have to be more precise in order to get $w < 0$.

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Hamiltonian approach

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Link between dissipativity expressed in Riemann coordinates and the Small Gain Theorem, using the Hamiltonian formulation

Valérie Dos Santos Martins[1]

***Abstract*— This paper aims at providing some synthesis between two alternative representations of systems of two conservation laws and to make a link between the notions of dissipativity and the Small-Gain theorem. The first one, based on the invariance of its coordinates, is the representation in Riemann coordinates which has been applied successfully for the stabilization of linear and non-linear hyperbolic systems of conservation laws. The second representation is based on physical modeling and leads to port Hamiltonian systems which are extensions of infinite-dimensional Hamiltonian systems defined on Dirac structure encompassing pairs of conjugated boundary variables [11].**

We propose in this paper to link the passivity of the Hamiltonian functional, expressed in Riemann coordinates, with the Small Gain Theorem.

I. Introduction

In this paper we shall be concerned with the stabilization via boundary control of hyperbolic systems of two conservation laws. The stabilization by boundary control of irrigation channels has been intensively studied for instance in [3], [9], [10] for both linear and non linear cases. The stability of hyperbolic partial differential equations on a one-dimensional spatial domain is widely studied in the literature [2], [4]. One of the most often suggested approaches, uses Riemann invariants to derive a stabilizing boundary control [12]. In recent publications, some extensions are suggested and based on the suitable choice of control Lyapunov function expressed in terms of Riemann's coordinates of the system [2], [4], [6], [7].

The use of physically motivated control Lyapunov function for the derivation of stabilizing control laws for non-linear finite-dimensional systems has proven to be very efficient and has lead to a great variety of results [5], [21], [22]. Very often, when the system stems from physical modeling, one may derive dissipation inequalities related to energy balance equations and energy dissipating phenomena [28]. Using the dissipative port-Hamiltonian formulation for controlled physical systems [5], [19], [22], one may go one step further and assign in closed loop not only some dissipation inequality for some suitable control Lyapunov function but also assign the dynamic behavior by the structure matrices of the Hamiltonian system in closed loop [18], [20]. For infinite-dimensional systems very similar techniques, based on dissipation inequalities, which in terms of PDE's amounts to consider some entropy function [2], [7], have been used for the stabilization of boundary control systems [8], [15]. Recent works have used a boundary port-Hamiltonian formulation of systems of conservation laws [17], [23] in order to derive stabilizing boundary control for a class of linear systems defined on one-dimensional spatial domains [14], [16], [26], [27].

The notion of dissipativity is generally linked with the Small-Gain Theorem. This link has been mentioned (e.g. [2], [3], [6], [7]) several times, but not proved for Greenberg & Li's theorem [12] and the generalized version of this theorem given by [7]. Some recents papers [24](more recently, works of H. Zwart) tends to put in light the Cayley transformation for PDE's control problems.

The sketch of the paper is the following. In a first instance the port Hamiltonian formulation is recalled with respect to a Stokes-Dirac structure [17], [23] in Riemann's coordinates, like the conditions on the boundary feedback relations derived with respect to the Riemann invariants; they are expressed in terms of the port boundary variables of the Hamiltonian formulation and interpreted in terms of the dissipation inequality of the Hamiltonian functional. The links described in Fig. 1 are then established in the third part.

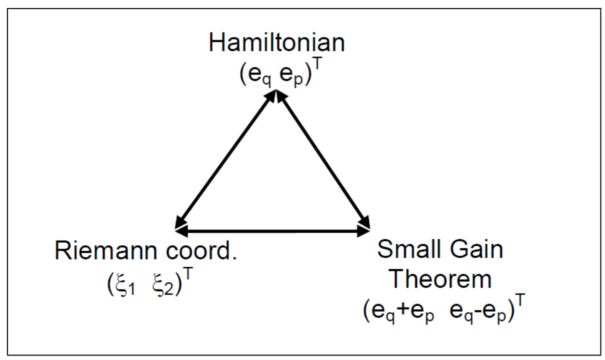

Fig. 1. Relations between Hamiltonian-Riemann-Small Gain Theorem

II. Port Hamiltonian formulation of a hyperbolic system of two conservation laws

A. Recall of the Riemann invariants for an hyperbolic system

In this section, we shall very briefly recall the main result on the stabilization of a hyperbolic system of two conservation laws suggested by Greenberg & Li [12]. Consider a spatial domain consisting of the finite interval $[0, L] \ni x$ with $L \in I\!R_+^*$ and time domain being the real interval $[0, +\infty) \ni t$. The state space is a non-empty connected

*This work was initiated in collaboration with B. Maschke and Y. Le Gorrec

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open set in $\mathbb{R}^2$, denoted by Ω. Consider the system of two conservations laws:

$$\partial_t Y + \partial_x f(Y) = 0, \tag{1}$$

where

- $Y = (y_1 \;\; y_2)^T : [0, +\infty) \times [0, L] \to \Omega$ is the vector of the two dependent variables;
- $f : \Omega \to \mathbb{R}^2$ is a C^1-function called *the flux vector*.

Note that the system (1) may also be written:

$$\partial_t Y + F(Y)\partial_x Y = 0 \tag{2}$$

where F is the Jacobian of the flux vector f. Assuming that the system is hyperbolic, implies that this system can be diagonalised using the Riemann invariants (see for instance [13, pages 34 - 35]). This means that there exists a change of coordinates $\xi(Y)$ whose Jacobian matrix is denoted $D(Y)$,

$$D(Y) = \frac{\partial \xi}{\partial Y}, \tag{3}$$

and diagonalises $F(Y)$ in Ω:

$$D(Y)F(Y) = \Lambda(Y)D(Y) \qquad\quad Y \in \Omega.$$

In the coordinates ξ, the system (1) can then be rewritten in the following (diagonal) characteristic form:

$$\partial_t \xi + \Lambda(\xi)\partial_x \xi = 0 \tag{4}$$

with ξ: $[0, L] \times [0, +\infty) \to \mathbb{R}^2$, $(x, t) \mapsto \xi(x, t)$, and $\Lambda(\xi) = \operatorname{diag}(\lambda_1(\xi), \lambda_2(\xi))$, with $\lambda_1(\xi), \lambda_2(\xi)$ satisfying the conditions:

- the λ_i's are continuously differentiable functions on a neighborhood of the origin;
- $\lambda_2(0) < 0 < \lambda_1(0)$.

In this paper we shall consider the following result of Greenberg and Li [12] which may be recalled as follows.

Theorem 2.1: Consider the hyperbolic system of conservation laws in Riemannian coordinates (4) with the following relations on the boundary variables:

$$\xi_2(0) = \mathbf{K_1}(\xi_2(0)),\ \xi_1(L) = \mathbf{K_2}(\xi_2(L)) \tag{5}$$

with the functions K_1 and K_2 being C^1 and satisfying:

$$\mathbf{K_1}(0) = \mathbf{K_2}(0) = 0 \text{ and } |\mathbf{K_1}'(0)\mathbf{K_2}'(0)| < 1. \tag{6}$$

Consider initial values:

$$\lim_{t \to 0^+} (\xi_1, \xi_2)(x, t) = (\xi_{1,0}, \xi_{2,0})(x),\ 0 < x < L, \tag{7}$$

being C^1 and satisfying the assumption that to be small in the C^1 norm and the compatibility conditions:

$$\xi_{2,0}(0) = \mathbf{K_1}(\xi_{1,0}(0)) \tag{8}$$

$$\xi_{1,0}(L) = \mathbf{K_2}(\xi_{2,0}(L)) \tag{9}$$

$$\begin{aligned}&\lambda_2(\xi_{1,0}, \xi_{2,0})(0)\, \partial_x \xi_{2,0}(0)\\ &\quad = \lambda_1(\xi_{1,0}, \xi_{2,0})(0)\, \mathbf{K_1}'(0)\, \partial_x \xi_{1,0}(0)\end{aligned} \tag{10}$$

$$\begin{aligned}&\lambda_1(\xi_{1,0}, \xi_{2,0})(L)\ \partial_x \xi_{1,0}(L)\\ &\quad = \lambda_2(\xi_{1,0}, \xi_{2,0})(L)\, \mathbf{K_2}'(L)\partial_x \xi_{2,0}(L)\end{aligned} \tag{11}$$

Then the initial value problem, for this system, has a unique C^1 solution. Moreover, its solution decays to zero in the C^1 norm with an exponential rate.

B. Boundary port Hamiltonian systems and Riemann coordinates

1) Hamiltonian operator expressed in the Riemann coordinates: In this section we shall consider a hyperbolic system of two conservation laws (1) which admits a Hamiltonian representation, that is such that vector of flux variables may be written following

$$\partial_x\, e(Y) = \mathcal{J} \begin{pmatrix} \delta_{y_1} H \\ \delta_{y_2} H \end{pmatrix}. \tag{12}$$

with the canonical Hamiltonian operator

$$\mathcal{J} = \epsilon \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{13}$$

Hence the system is written:

$$-\partial_t Y = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} (\delta_Y H). \tag{14}$$

In the sequel we express the Hamiltonian system in terms of the Rieman's invariants and give the expression of the Hamiltonian operator as well as of the boundary port variables. Denote by $\tilde{H}(\xi)$ the Hamiltonian expressed in the Riemann invariants: $\tilde{H}(\xi) = H \circ Y(\xi)$ where Y denotes, with an abuse of notation, the inverse change of coordinates to the Riemann coordinates.

Let us define $D_\xi = D \circ Y(\xi)$. One obtains by multiplying both terms of (14) by D:

$$\begin{aligned}
& -D(Y)\partial_t Y = D(Y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(\delta_Y H) \\
\Leftrightarrow\quad & -\partial_t \xi = D_\xi(\xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D^T(\xi)\delta_\xi \tilde{H}(\xi)) \\
\Leftrightarrow\quad & -\partial_t \xi = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D_\xi^T)\delta_\xi \tilde{H}(\xi) \\
& +D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\xi^T \partial_x(\delta_\xi \tilde{H}(\xi)).
\end{aligned}$$

Hence in terms of the Riemann invariants the system is written:

$$-\partial_t \xi = (B\partial_x + C)\delta_\xi \tilde{H}(\xi), \tag{15}$$

where $B = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_\xi^T$ and $C = D_\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x(D_\xi^T)$. The following properties may be noted: firstly the matrix B is symmetric and secondly it is related with the matrix C by:

$$\partial_x B = C^T + C. \tag{16}$$

2) Boundary port variables: In this section we shall check the formal skew-symmetry of the differential operator $(B\partial_x + C)$ and then define port boundary variables associated with it. Let us define the following bracket on smooth functions on the spatial domain $[0,\, L]$:

$$\{e_1, e_2\} = \int_0^L e_1^T (B\partial_x + C) e_2 \, dx \tag{17}$$

and consider the symmetric product [11]:

$$\{e_1, e_2\} + \{e_2, e_1\} = \int_0^L \partial_x (e_1^T B e_2) = \left[(e_1^T B e_2)\right]_0^L . \tag{18}$$

The product (18) corresponds to Stokes theorem applied to the equation for the differential operator $(B\partial_x + C)$. Furthermore the second member of (18) vanishes for all functions e_1, e_2 with compact support strictly included in the domain $[0,\, L]$ and hence for these functions the bracket is skew-symmetric.

However considering functions which do not vanish on the boundary of the domain, the bracket (17) is no more skew-symmetric. In this case the time variation of the Hamiltonian becomes:

$$\frac{d\tilde{H}(\xi)}{dt} = \left[(\delta_{\xi_1} \tilde{H}^T B \, \delta_{\xi_2} \tilde{H})\right]_0^L . \tag{19}$$

The definition of the boundary port variables follows strictly the construction suggested in [14]. Now the differential operator $B\partial_x + C$ completed with the definition defines the following vector space:

$$\tilde{\mathcal{D}} = \left\{ \left(\begin{pmatrix} f \\ f_\partial \end{pmatrix}, \begin{pmatrix} e \\ e_\partial \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} \;/\; f = (B\partial_x + C) e \right.$$
$$\left. \begin{pmatrix} e_\partial(L) \\ f_\partial(L) \\ e_\partial(0) \\ f_\partial(0) \end{pmatrix} = \mathbf{diag}\,(1,\, 1,\, -1,\, 1) \begin{pmatrix} e_1(L) \\ e_2(L) \\ e_1(0) \\ e_2(0) \end{pmatrix} \right\} \tag{20}$$

Adapting the proofs in [14], one may prove that the vector space $\tilde{\mathcal{D}}$ is a Dirac structure with respect to the pairing defined on $\left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \times \left(C^\infty[0,L] \times C^\infty[0,L] \times I\!R^2\right) \ni ((f,\, f_\partial), (e,\, e_\partial))$:

$$\langle (f,\, f_\partial), (e,\, e_\partial) \rangle = \int_0^L e^T f \, dx + e_\partial^T f_\partial$$

which is canonical in the sense that it does not depend on the differential operator anymore.

C. Stabilizing boundary relations with respect Riemann invariants and boundary port variables:

Consider a hyperbolic system of two conservation laws (1) which admits a Hamiltonian representation (14) with port variables.

Then, let us consider the relations on the boundary port variables defined by a C^1 function G:

$$\begin{pmatrix} e_\partial^0 \\ f_\partial^L \end{pmatrix} = G(f_\partial^0, e_\partial^L), \tag{21}$$

The energy balance equation depends on G_1, G_2 the line components of G and becomes:

$$\frac{dH}{dt} = e_\partial^L G_2(f_\partial^0, e_\partial^L) + f_\partial^0 G_1(f_\partial^0, e_\partial^L) \tag{22}$$

Using the implicit function theorem, the relations (21) on the port boundary variables, may be expressed in terms of boundary port variables (5) on the boundary values of the Riemann coordinates:

$$\begin{pmatrix} \xi_2'(0) \\ \xi_1'(L) \end{pmatrix} = \nabla K \begin{pmatrix} \xi_1'(0) \\ \xi_2'(L) \end{pmatrix}.$$

1) An abuse of notation is done all along the article in order to make easier the reading. According to the cases, the notation "f'" can stand for the derivative of f or its gradient.
2) The following notations are used,
$$\bar{G}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G'$$
and in the same idea
$$\hat{K}' = K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{23}$$
3) Let us pose F the jacobian and D the matrix of changes such that
$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
then
$$DF \;=\; \Lambda D \Leftrightarrow D = \begin{pmatrix} a & \frac{-a(\alpha - \lambda_1)}{\gamma} \\ c & \frac{-c(\alpha - \lambda_2)}{\gamma} \end{pmatrix} \tag{24}$$

Finally let us note that, using the derivative of the functions K_i of the boundary conditions, (5) can be expressed as:

$$\nabla \hat{K} \;=\; [Id - \mathcal{A}]^{-1} [Id + \mathcal{A}] \tag{25}$$

with
$$\mathcal{A} = \begin{pmatrix} \frac{2a}{\lambda_1}(0) & 0 \\ 0 & \frac{c(\lambda_2 - \lambda_1)}{\gamma \lambda_2}(L) \end{pmatrix} \nabla \bar{G} \begin{pmatrix} \frac{\gamma \lambda_1}{-a(\lambda_2 - \lambda_1)}(0) & 0 \\ 0 & \frac{\lambda_2}{2c}(L) \end{pmatrix}$$
under the condition that $[Id - \mathcal{A}]$ is invertible.

Proposition 1: If the components of $\nabla \bar{G}$ satisfies the following conditions

$$tr(\mathcal{A}) = -G_{21}' \frac{(\lambda_2 - \lambda_1)}{2\gamma}(L) + G_{12}' \frac{2\gamma}{(\lambda_2 - \lambda_1)}(0) < 0,$$

$G(0) = 0$ and
$$\det \mathcal{A} > 0$$
then the spectral radius of $[Id - \mathcal{A}]^{-1} [Id + \mathcal{A}]$ satisfy
$$\rho([Id - \mathcal{A}]^{-1} [Id + \mathcal{A}]) < 1$$
as it is exactly a Cayley transformation of the matrix if $\mathcal{A}$ is a closed operator [1].
Note that $\mathcal{A}$ is closed if $\nabla \bar{G}$ is invertible (not the only case). If furthermore the compatibility conditions (10)-(11) are satisfied the conditions of theorem 2.1 are satisfied and

the system is exponentially stable (see extension of this theorem given by [7]).

Proof: The proof is under review, and it is not the purpose of this article. An idea is given in [11]. Let remark that the sign of $\det \nabla G$ is the same that $\det \mathcal{A}$. ■

III. Link with the Small Gain Theorem

This notion of dissipativity is generally linked with the Small-Gain Theorem. This link has been mentioned (e.g. [7]) but not proved for Greenberg & Li's theorem (2.1) and the Riemann invariants in general.

It is proposed here to link the Hamiltonian functional expressed in Riemann coordinates with the Small Gain Theorem. Indeed, one can write:

$$d_t\tilde{H}_\xi = \begin{pmatrix}\delta_\xi H^T(L) & \delta_\xi H^T(0)\end{pmatrix}\begin{pmatrix}B(L) & 0\\ 0 & -B(0)\end{pmatrix}\begin{pmatrix}\delta_\xi H(L)\\ \delta_\xi H(0)\end{pmatrix} \quad (26)$$

$$= \frac{1}{2}\begin{pmatrix}e_2(L)+e_1(L)\\ e_2(0)-e_1(0)\\ e_2(L)-e_1(L)\\ e_2(0)+e_1(0)\end{pmatrix}^T \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}e_2(L)+e_1(L)\\ e_2(0)-e_1(0)\\ e_2(L)-e_1(L)\\ e_2(0)+e_1(0)\end{pmatrix} \quad (27)$$

Let define

$$y = \begin{pmatrix}e_2(L)+e_1(L)\\ e_2(0)-e_1(0)\end{pmatrix} \text{ and } u = \begin{pmatrix}e_2(L)-e_1(L)\\ e_2(0)+e_1(0)\end{pmatrix}$$

then

$$\begin{aligned} d_t\tilde{H}_\xi &= \frac{1}{2}\begin{pmatrix}y^T & u^T\end{pmatrix}^T\begin{pmatrix}I_2 & 0_2\\ 0_2 & -I_2\end{pmatrix}\begin{pmatrix}y\\ u\end{pmatrix} \\ &= \frac{1}{2}y^T y - \frac{1}{2}u^T u && (28)\\ \text{if} \quad & y = Mu && (29)\\ \text{then } d_t\tilde{H}_\xi &= \frac{1}{2}u^T[M^2-1]u && (30)\end{aligned}$$

Proposition 2: The Small Gain Theorem allows to conclude that the system (30) is stable if $\|M\| < 1$.

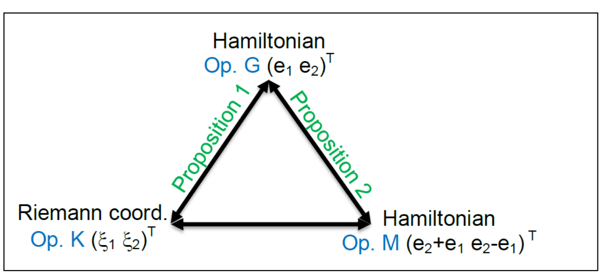

Fig. 2. Relations between Hamiltonian-Riemann-Small Gain Theorem

The main result of this paper can be stated:

Proposition 3: If the operator K established in (5) satisfies the generalized theorem (2.1) of Greenberg & Li defined in [7] then the operator M defined in (29) satisfies the condition $\|M\| < 1$ of the Small Gain Theorem.

Proof: As for the expression (25), the relation between operators M and G can be expressed as follow:

$$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} M = \left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\nabla G - I_2\right]\left[I_2 + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\nabla G\right]^{-1} \quad (31)$$

and $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} M$ is a Cayley transformation of $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\nabla G$.

Greenberg & Li theorem implies that the spectral radius of ∇K is less than 1. Given to the proposition 1, this implies that $det \nabla G > 0$ and $G'_{21} - G'_{12} < 0$, so

$$tr\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\nabla G\right) > 0 \quad (32)$$

$$\det\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\nabla G\right) = det\left(\nabla G\right) > 0 \quad (33)$$

By definition of the Cayley transformation, and as ∇G is invertible, the norm of $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} M$ is less than one, and so, the norm of M is less than one, too. The condition of the Small-Gain Theorem is satisfied. Let us remark that the norm is the L^2 norm. Another norm could be chosen as the theorem proved in [7] takes the $\sup$ of all the norms. ■

IV. Conclusion

In the first part of this paper we have recalled the expression of the port Hamiltonian systems of two conservation laws and derived its expression in terms of Riemann's invariant. In these coordinates, the expression of the Dirac structure associated with this expression of the matrix differential operator, the Hamiltonian operator is derived. The expression of the boundary port variables of the Hamiltonian formulation has also been derived. In the second part of the paper, the stability conditions on the boundary values of the Riemann invariants, with some conditions on the boundary constraints on the port variables, are expressed. As a consequence we have given an interpretation of the stabilizing boundary relations in terms of the dissipation of the Hamiltonian function on the boundary of the system. The third part, the Small-Gain theorem is applied to the Hamiltionian functional. The link between the dissipativity given in Riemann's coordinates and the condition issued from the Small-Gain theorem is established in one sense.

The futurs works would to define the conditions in order to prove the inverse relation going from the Small-Gain theorem, to the dissipativity relations.

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Some Properties of Conservative Port Contact Systems

Audrey Favache, Valérie Dos Santos, Denis Dochain, Bernhard Maschke

Abstract

The dynamics of open irreversible thermodynamic systems, that is systems including both the balance equation of the energy and the entropy, has been formulated as contact vector fields with generating functions depending on some external (control) variable and called conservative port contact systems. In this paper we relate the dynamical properties of these systems (equilibrium points, asymptotic stability) to properties of the generating functions (the contact Hamiltonian functions). We show that the equilibrium points of the system satisfy certain conditions involving the contact Hamiltonian function. We also consider Lyapunov's first theorem to emphasize a stability criterion for the equilibrium points in terms of this contact Hamiltonian function and relate it to some thermodynamical properties. These results are then related to the physical phenomena that are taking place in the system.

Index Terms

contact structure, irreversible thermodynamics, nonlinear dynamical systems

I. Introduction

Although many mathematical tools exist for analyzing the behaviour of nonlinear dynamical systems (including chemical systems), the link with the physical meaning is often not obvious.

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Indeed, these theories usually need to introduce abstract concepts like Lyapunov functions or storage functions, see for instance [1][2]. Nevertheless, for most systems these abstract concepts can be related to physical features. This is in particular the case for electromechanical systems through their Lagrangian or Hamiltonian formulation. For the past decade a new approach for the analysis and control of physical systems has received a growing interest. This approach is based on the physics of the systems. It uses the relation between the physics and the mathematical tools to develop control systems based on physics, such as for instance energy-shaping control [3][4][5][6][7][8] or power-shaping control [9][10][11][12]. One of the main interests of this approach is that it allows an easy handling of interconnected systems. This approach has also led to a generalization to the so-called port-Hamiltonian systems that allows to express the dynamics of systems implying reversible phenomena into a formalism which endows the state space with geometric structures (e.g. presymplectic, pseudo-Poisson or Dirac structures) [13][14].

However Poisson and Dirac structures apply in the first instance to models of reversible physical systems in the sense of thermodynamics, mainly lossless elasto-dynamic and electro-magnetic systems. When irreversible phenomena (that is some dissipation for instance through friction or Ohm's law) occur, the port-Hamiltonian system has to be augmented with a dissipative relation acting on some pair of port variables: the dissipation is treated as *external* to the system. Elimination of these variables leads to the definition of dissipative port Hamiltonian systems [15] defined with respect to a so-called Leibniz bracket [16]. In this case the Hamiltonian function is no more conserved and obeys a dissipation equality with a non-zero dissipation rate [17]: the Hamiltonian function does not represent anymore the total energy of the system but some of its Legendre transform. However it might be important to represent simultaneously the energy balance equation and the irreversibility, expressed through the entropy balance equation as it is the use in chemical engineering for instance: processes implying heat exchanges, mass transport or chemical reactions. Therefore another formalism has been recently developed that at the same time accounts for the energy and the entropy balance equations and generalizes the previously mentioned port-Hamiltonian approach. This approach is in line with Gibbs' work [18] and the use of Pfaffian equations in order to describe the thermodynamic properties of matter [19][20] [21][22]. The Thermodynamic Phase Space is then shown to be endowed with a canonical geometric structure associated with Gibbs' equation and called contact structure [20][23][24]. The dynamical models of open physical systems subject to irreversible physical phenomena may then be described by contact vector fields generated by contact Hamiltonian functions depending also on some external variable and leaving invariant some Legendre submanifold describing the thermodynamic properties of the system [14][25][26]. Such systems have been called *conservative port contact systems*. More recently a formulation which is better adapted to chemical engineering and more generally to mass and heat transport systems, has been suggested [27] where the generating function of the thermodynamic properties is the entropy function and the contact Hamiltonian function may be identified with a virtual entropy variation associated to irreversible (and reversible) phenomena. It has also been shown that the latter formulation allows to express the dynamics of a complex system consisting of an interconnection of subsystems exchanging matter and energy in terms of compartmental systems.

Nevertheless, in order to use the contact formalism for process control considerations, it is important to know how the dynamical properties of the system are expressed in this formalism. The aim of this paper is to present how equilibrium points and Lyapunov's first stability theorem are expressed in terms of the contact formalism. Since this formalism is closely related to the geometric structure of thermodynamics, this analysis can then be used to physically interpret the existence and/or the stability conditions of the steady states. After a brief summary of the main features of the contact formalism in Section II, we shall present the expression of the dynamical properties in Section III. Some possible physical interpretation of these results are then provided in Section IV.

Notations: the following notations have been considered throughout this paper :

- all vectors (including the gradients of scalar functions) are column vectors;
- the transpose of a vector or matrix x is denoted as x^t;
- the derivative of a scalar function is expressed as follows : $\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_k}\right]^t$ with $x = [x_1, x_2, ..., x_k]^t$

II. The contact structure formalism

In this section we first briefly introduce the differential-geometric structure of thermodynamics. In this framework Gibbs' relation is associated to a contact form and the thermodynamic properties of a system can be expressed as a Legendre submanifold. Furthermore we recall how the dynamical behaviour of thermodynamic systems may be formulated as a contact vector field leaving the Legendre submanifold defining its thermodynamic properties invariant [14][28].

A. The thermodynamic model

Equilibrium thermodynamics deals with the properties of matter and has been formalized in geometric terms by Gibbs [18] and in differential-geometric terms by Carathéodory [19]. The thermodynamic properties of a thermodynamic system may also be expressed in terms of contact geometry [14][20][22][29][30].

Let us consider a thermodynamic system with γ degrees of freedom. The state of the system is situated in the *Thermodynamic Phase Space* $\mathcal{T}$ that is a $(2\gamma+1)$-dimensional manifold. It may be endowed with a geometrical structure described by a 1-form θ fulfilling the following condition :

$$\theta \wedge (d\theta)^{\gamma} \neq 0$$

Then θ is called a *contact form* (where $\wedge$ denotes the exterior product) and the manifold $(\mathcal{T},\theta)$ endowed with the contact form θ is called a *contact manifold*. Gibbs' relation that describes the relations between the state variables corresponds to the Pfaffian equation $\theta = 0$. The γ-dimensional solutions of this Pfaffian equation are called *Legendre submanifolds*. According to Darboux's theorem, there exists a local set of coordinates $(x_0, x_1, ..., x_{\gamma}, p_1, ..., p_{\gamma})$ such that θ can be expressed in the following way in a neighbourhood $\mathcal{V}$ of every point of $\mathcal{T}$ [31] :

$$\theta|_{\mathcal{V}} = dx_0 - p^t dx \tag{1}$$

where $x = col\,(x_1, ..., x_{\gamma})$, $p = col\,(p_1, ..., p_{\gamma})$ and $|_{\mathcal{V}}$ indicates the restriction to $\mathcal{V}$. In the set of local coordinates, each Legendre submanifold of $(\mathcal{T},\theta)$ can be generated by a *generating function* $F\,(x_I, p_J)$ where $I \cup J = \{1, ..., \gamma\}$ is a disjoint partition of the set of indices. The coordinates of each point of the Legendre submanifold fulfill then the following relations :

$$x_0 = F\,(x_I, p_J) - p_J^t \frac{\partial F}{\partial p_J}, \quad x_J = -\frac{\partial F}{\partial p_J}, \quad p_I = \frac{\partial F}{\partial x_I} \tag{2}$$

Gibbs' theory [18] for thermodynamic systems can be translated directly from differential geometry by using these tools. As an illustration let us consider an homogeneous single phase thermodynamic system containing N_c chemical species. The state of the system is described by $2\,(N_c+2)+1$ state variables: the internal energy U, the volume V, the entropy S, the quantity of each species n, the temperature T, the pressure P and the chemical potential of each species μ. Note that n and μ are N_c-dimensional vectors with an entry for each species.

According to Gibbs' phase theorem, this system has $\gamma = N_c + 2$ degrees of freedom. Hence the state variables cannot vary independently. They are related by Gibbs' relation, usually written as follows [18][32][33] :

$$dU - TdS + PdV - \mu^t dn = 0 \qquad (3)$$

Nevertheless, in this paper we shall use the so-called "entropy form" of Gibbs' relation that is obtained by dividing (3) by $-T$:

$$dS - \frac{1}{T}dU - \frac{P}{T}dV + \frac{\mu}{T}^t dn = 0 \qquad (4)$$

By comparing this expression with equation (1), one may note that Gibbs' relation endows the Thermodynamic Phase Space with a contact form expressed in the canonical coordinates. Moreover the admissible states (i.e. the states fulfilling Gibbs' relation) for the system are located on some Legendre submanifold $\mathcal{L}$ of the Thermodynamic Phase Space. Let us remark that the states have a physical meaning only if they lie on the Legendre submanifold. Hence on the Legendre submanifold, the canonical coordinates can be associated to the physical state variables as follows :

$$\left\{\begin{array}{ccc} x_0 & \overset{\mathcal{L}}{\equiv} & S \\ x & \overset{\mathcal{L}}{\equiv} & [U, V, n]^t \\ p & \overset{\mathcal{L}}{\equiv} & \left[\frac{1}{T}, \frac{P}{T}, -\frac{\mu}{T}\right]^t \end{array}\right. \qquad (5)$$

Let us also note that each of the thermodynamic models of matter (e.g. the ideal gas, the ideal liquid or the van der Waals gas) corresponds to a Legendre submanifold of the Thermodynamic Phase Space, but that each possible Legendre submanifold that solves Gibbs' equation does not correspond to a physically representative thermodynamic model.

Gibbs' relation (4) indicates that there exists a function of γ state variables so that the $\gamma + 1$ remaining state variables can be expressed in terms of partial derivatives of this function. This one is called the *fundamental relation* [32], [33]. For instance, the function that gives the entropy as a function of the volume, the internal energy and the quantity of each species is a fundamental relation of the system :

$$S = \tilde{S}\left(U, V, n\right) \qquad (6)$$

The entropy form of Gibbs' relation shows that the temperature, the pressure and the chemical potential of each species are given by the first order partial derivative of this function :

$$\frac{1}{T} = \frac{\partial \tilde{S}}{\partial U}, \quad \frac{P}{T} = \frac{\partial \tilde{S}}{\partial V}, \quad -\frac{\mu}{T} = \frac{\partial \tilde{S}}{\partial n}$$

This can be compared to the relations given in (2) : one can then observe that the fundamental relation is in fact a generating function of the Legendre submanifold with $I = \{1, ..., \gamma\}$ and $J = \emptyset$.

However the Legendre submanifold may be defined with respect to another set of coordinates. The corresponding generating function is obtained by performing a Legendre transform of the entropy function $\widetilde{S}(U, V, n)$. The obtained generating functions are called Massieu-Planck functions and can be related to other thermodynamic potentials such as for instance the Helmholtz free energy or the Gibbs' free energy.

B. The system dynamics

The dynamics of a physical system undergoing some reversible and irreversible phenomena may be expressed by means of a so-called contact vector field. Contact vector fields on contact manifolds are the analogue of Hamiltonian vector fields on symplectic manifolds. In the sequel we shall only give a brief description of the contact vector fields and their expression in canonical coordinates (see [20][24] for more details).

A unique vector field $\mathcal{X}_f$ can be associated to every scalar function $f \in \mathcal{C}^\infty(\mathcal{T})$ defined on the contact manifold $(\mathcal{T}, \theta)$ [20][24][25]. These particular vector fields define transformations that preserve the contact structure and are called *contact vector fields*. The contact vector field $\mathcal{X}_f$ is said to be generated by $f(x_0, x, p)$ and is expressed in local coordinates as follows :

$$\mathcal{X}_f(x_0, x, p) = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -p^t \\ 0 & 0 & -I_\gamma \\ p & I_\gamma & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_0} \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial p} \end{pmatrix} \tag{7}$$

where I_γ is the identity matrix of dimension γ. The function $f(x_0, x, p)$ is called the *contact Hamiltonian function* by analogy with the Hamiltonian vector field on symplectic manifolds. Note that there is a bijection between the contact vector fields and the smooth functions.

It can be shown in addition that this vector field is tangent to a given Legendre submanifold $\mathcal{L}$ if and only if its contact Hamiltonian function is equal to zero on $\mathcal{L}$, i.e. if and only if :

$$f(x_0, x, p | (x_0, x, p) \in \mathcal{L}) = 0$$

The image of the Legendre submanifold $\mathcal{L}$ by the transformation defined by the contact vector field $\mathcal{X}_f$ is then $\mathcal{L}$.

Such contact vector fields have been defined for thermodynamical systems subject to reversible [34] and irreversible transformations [28]. They have been extended to open systems which are not in equilibrium with their environment [13][14][25][27] by considering contact Hamiltonian functions which depend also on some external (input) variables and have then been called control contact systems.

Let us now consider the dynamical behaviour of some thermodynamic system. Under the assumption of a quasi-static process, at each time, the current state variables fulfill the fundamental relation of the thermodynamic model. In other words, the trajectory of the system remains on the Legendre submanifold which is consequently an invariant of the system dynamics. Therefore the dynamical evolution of the system can be represented by a contact vector field. Since the trajectory has to remain on the Legendre submanifold, the contact vector field must be tangent to it. Thus the contact Hamiltonian function that generates the contact vector field has to be equal to zero on the Legendre submanifold associated to the thermodynamic model.

Example 1. *The aim of this example is to illustrate that the dynamics of a thermodynamic system can effectively be represented by contact Hamiltonian functions. First we establish the dynamical model of the system by physical considerations. Then we propose a contact Hamiltonian function that manages to represent the dynamics in the contact formulation.*

Let us consider a cell containing an ideal gas. The cell is heated by the environment. The heat transfer rate is proportional to the difference between the environment temperature T_{env} *and the cell temperature* T*. The heat transfer coefficient is denoted by* κ*. From (5) we have the following correspondance relations if we restrict on the Legendre submanifold* $\mathcal{L}$ *representing the ideal gas :*

$$x_0 \overset{\mathcal{L}}{\equiv} S \quad x \overset{\mathcal{L}}{\equiv} U \quad p \overset{\mathcal{L}}{\equiv} \frac{1}{T}$$

A generating function of the Legendre submanifold $\mathcal{L}$ *is given by* $F(x)$ *defined as follows (with the partition of indices* $I = \{1\}$ *and* $J = \emptyset$*) :*

$$F(x) = \tilde{S}\left(U \middle| U \equiv x\right)$$

where $\tilde{S}(U)$ *is the fundamental relation expressing the entropy of an ideal gas as a function of its internal energy.*

The internal energy balance is obtained by considering the inlet and outlet energy fluxes. The following relation is then obtained :

$$\frac{dU}{dt} = \kappa \left(T_{env} - T\right) \tag{8}$$

Let us consider the following contact Hamiltonian function candidate :

$$f_1\left(x_0, x, p\right) = \kappa \left(\frac{\partial F}{\partial x} - p\right)\left(T_{env} - \frac{1}{p}\right) \tag{9}$$

By introducing (9) *into* (11) *and then restricting to the Legendre submanifold, one obtains :*

$$\frac{dU}{dt} = \left.\frac{dx}{dt}\right|_{\mathcal{L}} = -\left.\frac{\partial f_1}{\partial p}\right|_{\mathcal{L}} = \kappa \left.\left[\left(T_{env} - \frac{1}{p}\right) - \kappa\left(\frac{\partial F}{\partial x} - p\right)\left(\frac{1}{p^2}\right)\right]\right|_{\mathcal{L}} \tag{10}$$

The last term is equal to zero on the Legendre submanifold and (10) reduces to :

$$\frac{dU}{dt} = \kappa \left.\left[\left(T_{env} - \frac{1}{p}\right)\right]\right|_{\mathcal{L}} = \kappa\left(T - T_{env}\right)$$

This is exactly the same expression as (8) *that was derived previously from physical considerations. This means that* (9) *can effectively represent the dynamical behaviour of the system.*

In summary, the system dynamics of a quasi-static thermodynamic process can be represented by the following relation :

$$\begin{cases} \frac{dx_0}{dt} & = & f(x_0, x, p) - p^t \frac{\partial f}{\partial p} \\ \frac{dx}{dt} & = & -\frac{\partial f}{\partial p} \\ \frac{dp}{dt} & = & \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial x_0} \end{cases} \tag{11}$$

where $f(x_0, x, p)$ is the contact Hamiltonian with $f\left(x_0, x, p\right) = 0$ on the Legendre submanifold representing the thermodynamic model. The only states (x_0, x, p) corresponding to a physically admissible state are located on the Legendre submanifold associated to the thermodynamic model and generated by the real-valued function $F(x_I, p_J)$ where $I \cup J = \{1, ..., \gamma\}$ is a disjoint partition of the set of indices.

III. Expression of the dynamical properties

A. Equivalent contact Hamiltonian functions

For systems describing mass and heat transport phenomena coupled with chemical reactions, as they arise in process systems, a family of contact Hamiltonian functions has been suggested in [27]. They have been derived starting from the expression of the flux variables in the balance equations of the internal energy, the volume and the quantity of each component. It has been shown that there can be several equivalent contact Hamiltonian functions to describe the same thermodynamic system in the sense that they generate contact vector fields which are equal on the Legendre submanifold characterizing the thermodynamic properties of the system. This comes from the fact that the contact vector field is obtained by lifting a vector field defined only on the Legendre submanifold (in this case, the balance equations on the extensive variables) to a contact vector field on the whole Thermodynamic Phase Space [14] and that such a lift may be not uniquely defined. In this section we shall characterize the set of the contact Hamiltonian functions that give rise to the same dynamical behaviour on a given Legendre submanifold.

Definition 1. *Consider a Legendre submanifold $\mathcal{L}$ of the contact manifold $(\mathcal{T}, \theta)$. Let f_1 and f_2 be two contact Hamiltonian functions that are equal to zero on $\mathcal{L}$. f_1 and f_2 are said to be* equivalent *on $\mathcal{L}$ if the generated contact vector fields $\mathcal{X}_{f_1}$ and $\mathcal{X}_{f_2}$ are equal on $\mathcal{L}$, i.e. $\mathcal{X}_{f_1}|_{\mathcal{L}} = \mathcal{X}_{f_2}|_{\mathcal{L}}$.*

The two following propositions show how to construct some families of contact Hamiltonian functions that are equivalent on $\mathcal{L}$.

Proposition 1. *Let us consider a Legendre submanifold $\mathcal{L}$ generated by the function $F(x_I, p_J)$ in the canonical coordinates as stated in relation (2), where $I \cup J$ is a disjoint partition of the set of indices. Let $f_1(x_0, x, p)$ be a contact Hamiltonian function that is equal to zero on $\mathcal{L}$. There always exist at least one smooth real-valued function $f_0(x_0, x, p)$ and one smooth real-valued function $\tilde{f}_1(x_0 x, p)$ such that $f_1(x_0, x, p)$ can be decomposed as follows :*

$$f_1(x_0, x, p) = f_0(x_0, x, p)\ \tilde{f}_1(x_0, x, p)$$

where $f_0(x_0, x, p) = 0$ on $\mathcal{L}$ and $\tilde{f}_1(x_0, x, p) \not\equiv 0$.

Consider the two following alternative contact Hamiltonian functions :

$$f_2(x_0, x, p) = f_0\left(x_0, x, p\right)\ \underbrace{\tilde{f}_1\left(x_0, x, p \middle|\ x_K = -\frac{\partial F}{\partial p_K}, p_L = \frac{\partial F}{\partial x_L}\right)}_{\tilde{f}_2(x_0, x_I, x_{J\setminus K}, p_J, p_{I\setminus L})}$$

$$f_3(x_0, x, p) = f_0\left(x_0, x, p\right)\ \underbrace{\tilde{f}_1\left(x_0, x, p \middle|\ x_0 = F(x_I, p_J) - p_J^t \frac{\partial F}{\partial p_J}\right)}_{\tilde{f}_3(x,p)}$$

where K and L are subsets of respectively J and I.

The restriction of the contact vector fields generated by the functions $f_1(x_0,x,p)$, $f_2(x_0,x,p)$ and $f_3(x_0,x,p)$ to the Legendre submanifold $\mathcal{L}$ are the same, i.e. $f_1(x_0,x,p)$, $f_2(x_0,x,p)$ and $f_3(x_0,x,p)$ are equivalent contact Hamiltonian functions on $\mathcal{L}$.

Proof: Since $f_0(x_0,x,p)=0$ on $\mathcal{L}$, it is straightforward to see that $f_2(x_0,x,p)$ and $f_3(x_0,x,p)$ are equal to zero on $\mathcal{L}$. Consequently the vector fields generated by $f_2(x_0,x,p)$ and $f_3(x_0,x,p)$ leave the Legendre submanifold $\mathcal{L}$ invariant. Moreover, by using (2), it can be observed that :

$$\tilde{f}_1(x_0,x,p)\Big|_{\mathcal{L}} = \tilde{f}_2(x_0,x,p)\Big|_{\mathcal{L}} = \tilde{f}_3(x_0,x,p)\Big|_{\mathcal{L}} \tag{12}$$

Let us now write in canonical coordinates the expression of the generated vector fields by using (7) :

$$\mathcal{X}_{f_i} = \begin{pmatrix} f_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -p^t \\ 0 & 0 & -I_\gamma \\ p & I_\gamma & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_0}{\partial x_0}\tilde{f}_i + f_0\frac{\partial \tilde{f}_i}{\partial x_0} \\ \frac{\partial f_0}{\partial x}\tilde{f}_i + f_0\frac{\partial \tilde{f}_i}{\partial x} \\ \frac{\partial f_0}{\partial p}\tilde{f}_i + f_0\frac{\partial \tilde{f}_i}{\partial p} \end{pmatrix} \overset{\mathcal{L}}{=} \begin{pmatrix} 0 & 0 & -p^t \\ 0 & 0 & -I_\gamma \\ p & I_\gamma & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_0}{\partial x_0} \\ \frac{\partial f_0}{\partial x} \\ \frac{\partial f_0}{\partial p} \end{pmatrix} \tilde{f}_i\Big|_{\mathcal{L}}$$

where $i=1,2,3$. By using (12), one immediately obtains :

$$\mathcal{X}_{f_1}|_{\mathcal{L}} = \mathcal{X}_{f_2}|_{\mathcal{L}} = \mathcal{X}_{f_3}|_{\mathcal{L}}$$

■

Another interesting property of equivalent contact Hamiltonian functions is that they can be combined to find another equivalent contact Hamiltonian function. This is stated in the following proposition.

Proposition 2. *Let $\mathcal{L}$ be a Legendre submanifold of the contact manifold $(\mathcal{T},\theta)$. Let $f_1, f_2, ..., f_k$ be k equivalent contact Hamiltonian functions on $\mathcal{L}$ with :*

$$f_1|_{\mathcal{L}} = f_2|_{\mathcal{L}} = ... = f_k|_{\mathcal{L}} = 0$$

Let $\alpha_1, \alpha_2, ..., \alpha_k$ be k scalar numbers with $\sum_{i=1}^{k} \alpha_i = 1$. Then the following contact Hamiltonian function :

$$f = \sum_{i=1}^{k} \alpha_i f_i$$

is also an equivalent contact Hamiltonian function to $f_1, f_2, ..., f_k$ on $\mathcal{L}$ with $f|_{\mathcal{L}} = 0$.

Proof: It is straightforward that $f|_{\mathcal{L}} = 0$ since $f_1|_{\mathcal{L}} = f_2|_{\mathcal{L}} = ... = f_k|_{\mathcal{L}} = 0$. Let us now express the contact vector field generated by the function f expressed in the canonical coordinates using relation (7) :

$$\begin{aligned}\mathcal{X}_f &= \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -p^t \\ 0 & 0 & -I_\gamma \\ p & I_\gamma & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_0} \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial p} \end{pmatrix} \\ &= \sum_{i=1}^{k} \alpha_i \begin{pmatrix} f_i \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^{k} \alpha_i \begin{pmatrix} 0 & 0 & -p^t \\ 0 & 0 & -I_\gamma \\ p & I_\gamma & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_i}{\partial x_0} \\ \frac{\partial f_i}{\partial x} \\ \frac{\partial f_i}{\partial p} \end{pmatrix} = \sum_{i=1}^{k} \alpha_i \mathcal{X}_{f_i}\end{aligned}$$

Now let us restrict $\mathcal{X}_f$ to the Legendre submanifold. By using the fact that :

$$\mathcal{X}_{f_1}|_{\mathcal{L}} = \mathcal{X}_{f_2}|_{\mathcal{L}} = ... = \mathcal{X}_{f_k}|_{\mathcal{L}}$$

one obtains the following relation :

$$\mathcal{X}_f|_{\mathcal{L}} = \sum_{i=1}^{k} \alpha_i \, \mathcal{X}_{f_i}|_{\mathcal{L}} = \sum_{i=1}^{k} \alpha_i \, \mathcal{X}_{f_1}|_{\mathcal{L}} = \mathcal{X}_{f_1}|_{\mathcal{L}}$$

Consequently, following Definition 1, $f = \sum_{i=1}^{k} \alpha_i f_i$ is an equivalent contact Hamiltonian function to $f_1, f_2, ..., f_k$ on $\mathcal{L}$. ■

Definition 1 and Proposition 1 are illustrated by the following example.

Example 1 (continued). *Let us consider again the system of Example 1 with the contact Hamiltonian function* $f_1(x_0, x, p)$ *defined in* (9). *Now let us choose the functions* $f_0(x_0, x, p)$ *and* $\tilde{f}_1(x_0, x, p)$ *as follows :*

$$f_0(x_0, x, p) = \left(\frac{\partial F}{\partial x} - p\right), \; \tilde{f}_1(x_0, x, p) = \kappa \left(T_{env} - \frac{1}{p}\right)$$

It is straightforward to see that $f_0(x_0, x, p) = 0$ *on* $\mathcal{L}$ *and* $\tilde{f}_1(x_0, x, p) \neq 0$ *for* $(x_0, x, p) \in \mathcal{L} \setminus \left\{\left(x_0, x, p = \frac{1}{T_{env}}\right)\right\}$. *Therefore, following Proposition 1 the following contact Hamiltonian function* $f_2(x_0, x, p)$ *is equivalent on* $\mathcal{L}$ *to* $f_1(x_0, x, p)$ *:*

$$f_2(x_0, x, p) = f_0(x_0, x, p)\, \tilde{f}_1\left(x_0, x, \frac{\partial F}{\partial x}\right) = \left(\frac{\partial F}{\partial x} - p\right) \kappa \left(T_{env} - \left(\frac{\partial F}{\partial x}\right)^{-1}\right)$$

It can be easily checked out that the contact vector field generated by $f_2(x_0, x, p)$ *can also represent the dynamics of the system by redoing Example 1 with* $f_2(x_0, x, p)$ *instead of* $f_1(x_0, x, p)$.

B. Equilibrium points

Let us now derive the equilibrium points of the dynamical system and express them as particular points of the contact Hamiltonian function.

Proposition 3. *Consider a contact manifold $(\mathcal{T}, \theta)$ and a contact vector field $\mathcal{X}_f$. $\mathcal{X}_f$ is generated by the contact Hamiltonian function $f(x_0, x, p)$ given in the canonical coordinates (x_0, x, p). Let us consider a state $(\bar{x}_0, \bar{x}, \bar{p}) \in (\mathcal{T}, \theta)$. Then $(\bar{x}_0, \bar{x}, \bar{p}) \in \{(x_0, x, p)\,|\,\mathcal{X}_f(x_0, x, p) = 0\}$ if and only if the following conditions are fulfilled :*

$$\left.\frac{\partial f}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})} = 0 \tag{13a}$$

$$f(\bar{x}_0, \bar{x}, \bar{p}) = 0 \tag{13b}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})} = -\bar{p}\left.\frac{\partial f}{\partial x_0}\right|_{(\bar{x}_0,\bar{x},\bar{p})} \tag{13c}$$

Proof: Let us consider a point $(\bar{x}_0, \bar{x}, \bar{p}) \in (\mathcal{T}, \theta)$. $(\bar{x}_0, \bar{x}, \bar{p}) \in \{(x_0, x, p)\,|\,\mathcal{X}_f(x_0, x, p) = 0\}$ implies the following equality :

$$\mathcal{X}_f\left(\bar{x}_0, \bar{x}, \bar{p}\right) = \begin{pmatrix} f\left(\bar{x}_0, \bar{x}, \bar{p}\right) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\bar{p}^t \\ 0 & 0 & -I_\gamma \\ \bar{p} & I_\gamma & 0 \end{pmatrix} \begin{pmatrix} \left.\frac{\partial f}{\partial x_0}\right|_{(\bar{x}_0,\bar{x},\bar{p})} \\ \left.\frac{\partial f}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})} \\ \left.\frac{\partial f}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By developing the above expression, one obtains the following relations:

$$\begin{aligned} f\left(\bar{x}_0, \bar{x}, \bar{p}\right) - \bar{p}^t \left.\frac{\partial f}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})} &= 0 \\ -\left.\frac{\partial f}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})} &= 0 \\ \bar{p}\left.\frac{\partial f}{\partial x_0}\right|_{(\bar{x}_0,\bar{x},\bar{p})} + \left.\frac{\partial f}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})} &= 0 \end{aligned}$$

The second and third equations directly lead to the conditions (13a) and (13c), respectively. By introducing condition (13a) into the first equation, one obtains the condition (13b).

To prove the «only if» part, one only needs to check that if $(\bar{x}_0, \bar{x}, \bar{p})$ fulfills the conditions (13a) to (13c), then $\mathcal{X}_f\left(\bar{x}_0, \bar{x}, \bar{p}\right) = 0$ by considering the expression of $\mathcal{X}_f$ in canonical coordinates given by (7). ∎

Corollary 1. *Consider a contact manifold* $(\mathcal{T}, \theta)$ *and a contact vector field* $\mathcal{X}_f$ *generated by the contact Hamiltonian function* $f(x_0, x, p)$ *given in the canonical coordinates* (x_0, x, p)*. Assume that* $f(x_0, x, p)$ *does not depend on* x_0*. Let us consider a state* $(\bar{x}_0, \bar{x}, \bar{p}) \in (\mathcal{T}, \theta)$*. Then* $\mathcal{X}_f\left(\bar{x}_0, \bar{x}, \bar{p}\right) = 0$ *if and only if* $(\bar{x}_0, \bar{x}, \bar{p})$ *is a critical point and a zero of* $f(x_0, x, p)$*.*

Proof: It is obvious to check that if $f(x_0, x, p)$ does not depend on x_0, the conditions (13a) and (13c) become $\nabla f(\bar{x}_0, \bar{x}, \bar{p}) = 0$ (where ∇f denotes the gradient of the function f), which is equivalent to say that $(\bar{x}_0, \bar{x}, \bar{p})$ is a critical point of $f(x_0, x, p)$. Condition (13b) is equivalent to the fact that $(\bar{x}_0, \bar{x}, \bar{p})$ is a zero of $f(x_0, x, p)$. ■

Proposition 3 and Corollary 1 can be applied to find the equilibrium points of some physical process. Let us consider a system whose dynamical behaviour is expressed by a contact Hamiltonian function $f(x_0, x, p)$ using relation (11) where $f(x_0, x, p) = 0$ on some Legendre submanifold $\mathcal{L}$ of the Thermodynamic Phase Space. $\mathcal{L}$ represents the thermodynamic model of the system. The equilibria of the physical system are the points that fulfill the conditions (13a) to (13c) which are also located on the Legendre submanifold associated to the thermodynamic model. Moreover if $f(x_0, x, p)$ does not depend on x_0, then from Corollary 1, the equilibrium points of the physical system are all the critical points of $f(x_0, x, p)$ that are situated on $\mathcal{L}$.

Nevertheless, an equilibrium point on the Legendre submanifold can never be a local extremum of the contact Hamiltonian function. Indeed there exist points in the neighbourhood of the critical points for which the contact Hamiltonian function takes the same value than on the critical point, namely zero since the contact Hamiltonian function is invariant on the Legendre submanifold.

Example 1 (continued). *Let us consider Example 1 with the contact Hamiltonian function :*

$$f_1\left(x_0, x, p\right) = \kappa \left(\frac{\partial F}{\partial x} - p\right)\left(T_{env} - \frac{1}{p}\right)$$

where $F(x)$ *is the generating function in canonical coordinates of the Legendre submanifold representing the ideal gas for the following partition of the indices:* $I = \{1\}$ *and* $J = \emptyset$*.* $f_1\left(x_0, x, p\right)$ *does not depend on* x_0*. Therefore the equilibrium points* $(\bar{x},\bar{p})$ *are the points for which the gradient of the contact Hamiltonian function is equal to zero :*

$$\nabla f_1(\bar{x},\bar{p}) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x}\right|_{(\bar{x},\bar{p})} \\ \left.\frac{\partial f_1}{\partial p}\right|_{(\bar{x},\bar{p})} \end{pmatrix} = \begin{pmatrix} \kappa \left.\frac{\partial^2 F}{\partial x^2}\right|_{(\bar{x},\bar{p})} \left(T_{env} - \frac{1}{\bar{p}}\right) \\ \kappa(\frac{1}{\bar{p}} - T_{env}) + \kappa(\left.\frac{\partial F}{\partial x}\right|_{(\bar{x},\bar{p})} - \bar{p})(\frac{1}{\bar{p}^2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

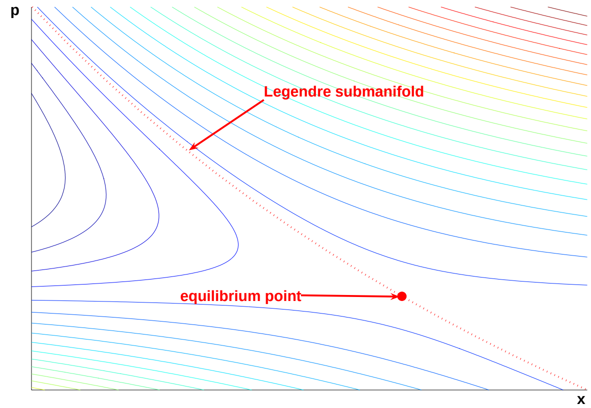

Figure 1. Contour plot of the contact Hamiltonian function for Example 1

The set Ω of critical points $(\bar{x}_0, \bar{x}, \bar{p})$ is therefore :

$$\Omega = \left\{ (\bar{x}_0, \bar{x}, \bar{p}) | \, \bar{p} = \frac{1}{T_{env}}, \bar{p} = \left. \frac{\partial F}{\partial x} \right|_{\bar{x}} \right\}$$

We are only interested in equilibrium points situated on $\mathcal{L}$, i.e. the set $\Omega \cap \mathcal{L}$:

$$\Omega \cap \mathcal{L} = \left\{ (\bar{x}_0, \bar{x}, \bar{p}) | \, \bar{p} = \frac{1}{T_{env}}, \bar{p} = \left. \frac{\partial F}{\partial x} \right|_{\bar{x}}, \bar{x}_0 = F(\bar{x}) \right\}$$

Since on the Legendre submanifold, the variable p is associated to the inverse of the temperature, the equilibrium points are all the states for which the temperature inside the cell is equal to the temperature outside. The result is presented graphically on Figure 1 where it can be seen that the equilibrium is a saddle point of the contact Hamiltonian function.

C. Sufficient stability condition

Let us now develop a sufficient condition for the asymptotic stability of an equilibrium point on the generating function of the Legendre submanifold and the contact Hamiltonian function. Indeed we shall express Lyapunov's first theorem in terms of these two functions.

Lyapunov's first theorem is based on the fact that for hyperbolic equilibrium points, the trajectories of the linearized system (around the equilibrium point) are topologically equivalent to those of the system. We shall first linearize the system dynamics when expressed in the contact structure formalism. Then we shall apply Lyapunov's first method in order to derive a sufficient stability criterion for a given equilibrium point. The result is stated in the following proposition.

Proposition 4. *Let $\mathcal{L}$ be a Legendre submanifold of the contact manifold $(\mathcal{T}, \theta)$. The generating function of $\mathcal{L}$ is $F(x_I, p_J)$ in the canonical coordinates, with $I \cup J$ a disjoint partition of the indices. Consider a system whose dynamical behaviour on $\mathcal{L}$ is given by the contact Hamiltonian function $f(x_0, x, p)$ with $f(x_0, x, p)|_{\mathcal{L}} = 0$. Let us assume that $(\bar{x}_0, \bar{x}, \bar{p}) \in \mathcal{L}$ is an equilibrium of the system. Let us define the matrix $\Lambda(x_0, x, p)$ given by following expression :*

$$
\begin{aligned}
\Lambda \;=\; & \left[\begin{pmatrix} -\frac{\partial^2 f}{\partial p_I^2} & (\frac{\partial^2 f}{\partial x_J \partial p_I})^t \\ \frac{\partial^2 f}{\partial x_J \partial p_I} & -\frac{\partial^2 f}{\partial x_J^2} \end{pmatrix} + \begin{pmatrix} 0 & (\frac{\partial^2 f}{\partial x_0 \partial p_I})^t p_J^t \\ p_J \frac{\partial^2 f}{\partial x_0 \partial p_I} & -\frac{\partial^2 f}{\partial x_0^2} p_J p_J^t - \frac{\partial^2 f}{\partial x_0 \partial x_J} p_J^t - p_J \frac{\partial^2 f}{\partial x_0 \partial p_J} \end{pmatrix} \right] \partial^2 F \\
& + \begin{pmatrix} -(\frac{\partial^2 f}{\partial x_0 \partial p_I})^t \frac{\partial^t F}{\partial x_I} & 0 \\ \frac{\partial^2 f}{\partial x_0^2} p_J \frac{\partial^t F}{\partial x_I} + \frac{\partial^2 f}{\partial x_0 \partial x_J} \frac{\partial^t F}{\partial x_I} + p_J \frac{\partial^2 f}{\partial x_0 \partial x_I} & \frac{\partial f}{\partial x_0} I + p_J \frac{\partial^2 f}{\partial x_0 \partial p_J} \end{pmatrix} \\
& + \begin{pmatrix} -\left(\frac{\partial^2 f}{\partial x_I \partial p_I}\right)^t & -\frac{\partial^2 f}{\partial p_I \partial p_J} \\ \left(\frac{\partial^2 f}{\partial x_I \partial x_J}\right)^t & \frac{\partial^2 f}{\partial x_J \partial p_J} \end{pmatrix}
\end{aligned}
$$

where $\partial^2 F = \begin{pmatrix} \frac{\partial^2 F}{\partial x_I^2} & \frac{\partial^2 F}{\partial x_I \partial p_J} \\ \left(\frac{\partial^2 F}{\partial x_I \partial p_J}\right)^t & \frac{\partial^2 F}{\partial p_J^2} \end{pmatrix}$ *and I is the identity matrix of adequate size.*

Then $(\bar{x}_0, \bar{x}, \bar{p})$ is a locally exponentially stable equilibrium of the dynamics restricted to $\mathcal{L}$ if the matrix $\Lambda(\bar{x}_0, \bar{x}, \bar{p})$ has eigenvalues with strictly negative real parts. If $\Lambda(\bar{x}_0, \bar{x}, \bar{p})$ has one or more eigenvalues with strictly positive real part, then $(\bar{x}_0, \bar{x}, \bar{p})$ is an unstable equilibrium point of the dynamics reduced to $\mathcal{L}$.

Proof: First let us linearize the system dynamics around the equilibrium point $(\bar{x}_0, \bar{x}, \bar{p})$. We are interested only in the dynamics restricted to $\mathcal{L}$. The time variation of the variables x_0, x_J and p_I are linked to the time variation of x_I and p_J since the trajectories remain on the Legendre submanifold $\mathcal{L}$. Therefore it is sufficient to linearize the following reduced dynamical system consisting in the time evolution of x_I and p_J :

$$
\frac{dx_I}{dt} \;=\; -\frac{\partial f}{\partial p_I} = z_x(x_0, x, p) \stackrel{\mathcal{L}}{=} \mathcal{Z}_x(x_I, p_J) \tag{15}
$$

$$
\frac{dp_J}{dt} \;=\; p_J \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_J} = z_p(x_0, x, p) \stackrel{\mathcal{L}}{=} \mathcal{Z}_p(x_I, p_J) \tag{16}
$$

The linearized dynamics of x_I is given by following expression :

$$\begin{aligned}
\frac{d\tilde{x}_I}{dt} &= \left(\frac{\partial^t \mathcal{Z}_x}{\partial x_I}\right)_{(\bar{x}_0,\bar{x},\bar{p})} \tilde{x}_I + \left(\frac{\partial^t \mathcal{Z}_p}{\partial x_I}\right)_{(\bar{x}_0,\bar{x},\bar{p})} \tilde{p}_J \\
&= \left[\frac{\partial^t z_x}{\partial x_0}\frac{\partial^t F}{\partial x_I} - \frac{\partial^t z_x}{\partial x_0} p_J^t \left(\frac{\partial^2 F}{\partial x_I \partial p_J}\right)^t \right. \\
&\qquad \left. - \frac{\partial^t z_x}{\partial x_J}\left(\frac{\partial^2 F}{\partial x_I \partial p_J}\right)^t + \frac{\partial^t z_x}{\partial x_I} + \frac{\partial^t z_x}{\partial p_I}\frac{\partial^2 F}{\partial x_I^2}\right]_{(\bar{x}_0,\bar{x},\bar{p})} \tilde{x}_I \\
&\quad + \left[-\frac{\partial^t z_x}{\partial x_0} p_J^t \left(\frac{\partial^2 F}{\partial x_I \partial p_J}\right)^t - \frac{\partial^t z_x}{\partial x_J}\left(\frac{\partial^2 F}{\partial p_J^2}\right)^t \right. \\
&\qquad \left. + \frac{\partial^t z_x}{\partial p_J} + \frac{\partial^t z_x}{\partial p_I}\frac{\partial^2 F}{\partial x_I \partial p_J}\right]_{(\bar{x}_0,\bar{x},\bar{p})} \tilde{p}_J
\end{aligned}$$

where $\tilde{x}_I = x_I - \bar{x}_I$ and $\tilde{p}_J = p_J - \bar{p}_J$. This result has been obtained by using the relations between the variables on $\mathcal{L}$ given by (2). The linearized dynamics of p_J is given by the same relation when replacing $z_x(x_0, x, p)$ by $z_p(x_0, x, p)$. $z_x(x_0, x, p)$ and $z_p(x_0, x, p)$ can now be replaced by their expression in terms of $f(x_0, x, p)$. For the sake of brevity and clarity, the calculation details are not given here. The result is the following expression for the linearized system :

$$\begin{pmatrix} \frac{d\tilde{x}_I}{dt} \\ \frac{d\tilde{p}_J}{dt} \end{pmatrix} = \Lambda(\bar{x}_0, \bar{x}, \bar{p}) \begin{pmatrix} \tilde{x}_I \\ \tilde{p}_J \end{pmatrix} \tag{17}$$

where the expression for $\Lambda(x_0, x, p)$ is given in Proposition 4.

Following Lyapunov's first method, the dynamical system (15)(16) has a locally exponentially stable equilibrium at $(\bar{x}_0, \bar{x}, \bar{p})$ if the linearized system (17) has an exponentially stable equilibrium at $(\bar{x}_0, \bar{x}, \bar{p})$, i.e. if $\Lambda(\bar{x}_0, \bar{x}, \bar{p})$ has eigenvalues with strictly negative real parts. If the linearized system (17) is unstable, i.e. if $\Lambda(\bar{x}_0, \bar{x}, \bar{p})$ has one or more eigenvalues with strictly positive real part, then $(\bar{x}_0, \bar{x}, \bar{p})$ is an unstable equilibrium point of the dynamical system (15)(16). ■

Corollary 2. *In addition to the assumptions of Proposition 4, let us assume that $f(x_0, x, p)$ does not depend on x_0. Then $(\bar{x}_0, \bar{x}, \bar{p})$ is a locally exponentially stable equilibrium of the dynamics restricted to $\mathcal{L}$ if the matrix $\Lambda(x, p)$ has eigenvalues with strictly negative real parts at $(\bar{x}_0, \bar{x}, \bar{p})$ where $\Lambda(x, p)$ is given by the following expression:*

$$\Lambda(x,p) = \begin{pmatrix} -\frac{\partial^2 f}{\partial p_I^2} & \left(\frac{\partial^2 f}{\partial x_J \partial p_I}\right)^t \\ \frac{\partial^2 f}{\partial x_J \partial p_I} & -\frac{\partial^2 f}{\partial x_J^2} \end{pmatrix} \partial^2 F + \begin{pmatrix} -\left(\frac{\partial^2 f}{\partial x_I \partial p_I}\right)^t & -\frac{\partial^2 f}{\partial p_I \partial p_J} \\ \left(\frac{\partial^2 f}{\partial x_I \partial x_J}\right)^t & \frac{\partial^2 f}{\partial x_J \partial p_J} \end{pmatrix} \tag{18}$$

where $\partial^2 F = \begin{pmatrix} \frac{\partial^2 F}{\partial x_I^2} & \frac{\partial^2 F}{\partial x_I \partial p_J} \\ \left(\frac{\partial^2 F}{\partial x_I \partial p_J}\right)^t & \frac{\partial^2 F}{\partial p_J^2} \end{pmatrix}$. *If* $\Lambda(\bar{x}, \bar{p})$ *has one or more eigenvalues with strictly positive real part, then* $(\bar{x}_0, \bar{x}, \bar{p})$ *is an unstable equilibrium point of the dynamics reduced to* $\mathcal{L}$.

Proof: straighforward by including the fact that $f(x_0, x, p)$ does not depend on x_0 in the expression of the matrix $\Lambda\,(x_0, x, p)$ in Proposition 4. ■

Example 1 (continued). *Let us consider Example 1 with the contact Hamiltonian function :*

$$f_1\,(x_0, x, p) = \kappa \left(\frac{\partial F}{\partial x} - p \right) \left(T_{env} - \frac{1}{p} \right)$$

where $F(x)$ *is the generating function in canonical coordinates of the Legendre submanifold representing the ideal gas for following partition of the indices:* $I = \{1\}$ *and* $J = \emptyset$. $f_1\,(x_0, x, p)$ *does not depend on* x_0. *The matrix* Λ *of Proposition 4 is given by relation (18) :*

$$\Lambda(x, p) = -\frac{\partial^2 f_1}{\partial p^2}\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 f_1}{\partial x \partial p} = \kappa \frac{\partial^2 F}{\partial x^2}\frac{1}{p^2} + \kappa \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial x} - p \right)\frac{2}{p^3}$$

The equilibrium point of the system is given by (see (14)) :

$$\bar{p} = \frac{1}{T_{env}} \quad \text{and} \quad \bar{p} = \left.\frac{\partial F}{\partial x}\right|_{\bar{x}}$$

Therefore at the equilibrium, the matrix Λ *becomes :*

$$\Lambda(\bar{x}, \bar{p}) = \kappa T_{env}^2 \left.\frac{\partial^2 F}{\partial x^2}\right|_{\bar{x}} \tag{19}$$

The equilibrium is exponentially stable if $F(x)$ *is concave and unstable if* $F(x)$ *is convex.*

In Example 1 it has been explained that our choice of the generating function is such that :

$$F\,(x) = \tilde{S}\,(U|\, U \equiv x)$$

where $\tilde{S}\,(U)$ *is the fundamental relation expressing the entropy of an ideal gas as a function of its internal energy. Hence,* (19) *can be written as follows :*

$$\Lambda(\bar{x}, \bar{p}) = \kappa T_{env}^2 \left.\frac{\partial^2 F}{\partial x^2}\right|_{\bar{x}} = \kappa T_{env}^2 \left.\frac{\partial^2 \tilde{S}}{\partial U^2}\right|_{\bar{U}}$$

A direct consequence of the second principle of thermodynamics is the concavity of the entropy function, i.e. $\frac{\partial^2 \tilde{S}}{\partial U^2}$ *is negative definite [32]. As a consequence, the concavity of the entropy implies the stability of the equilibrium.*

IV. Physical interpretation of the stability condition

In the above example, the stability condition of Proposition 4 has been related to the concavity of the entropy function. This suggests that the stability condition can possibly be related to the thermodynamic laws and therefore can find a physical interpretation. Let us now give some physical insight into the mathematical aspects developed in the previous section. In other words we shall link here in a general way the sufficient stability condition to the usual thermodynamic considerations on the system.

Let us consider the dynamical behaviour of some thermodynamic system whose state is given by $2\gamma + 1$ thermodynamic state variables. The thermodynamic model is represented by the Legendre submanifold $\mathcal{L}$ of the contact manifold $(\mathcal{T}, \theta)$ where $\mathcal{T}$ is the Thermodynamic Phase Space and θ is the Gibbs' relation. Without any loss of generality, we shall consider here the case where θ is the Gibbs' relation in its entropy form (4). The following considerations remain true for other forms of the Gibbs' relation by adapting the correspondence relations between the canonical coordinates and the thermodynamic state variables as they are given by (5) for the entropy form. In addition let us make the following assumptions :

Assumption 1. *The generating function of the Legendre submanifold is given for the partition of the indices $I = \{1, ..., \gamma\}$ and $J = \emptyset$ where γ is the dimension of the Legendre submanifold (i.e. the number of degrees of freedom of the system).*

In other words, the generating function of the Legendre submanifold is the entropy as a function of the extensive quantities, as it has been explained in Section II-A in (6). Consequently the coordinates of the points of $\mathcal{L}$ fulfill the following relations :

$$x_0 = \tilde{S}\left(x\right), \quad p = \frac{\partial \tilde{S}}{\partial x} \tag{20}$$

where

$$\tilde{S}\left(x\right) = \tilde{S}\left(x \middle| x = [U, V, n]^t\right)$$

is the fundamental relation of the thermodynamic model.

Assumption 2. *The system dynamics are given by the balance equations on the extensive quantities related to the x-coordinates (i.e. the internal energy, the volume and the quantity of each species when θ is the entropy form of Gibbs' relation).*

Assumption 3. *The net flowrates of the extensive quantities related to the x-coordinates do not depend on the quantity related to the x_0-coordinate (i.e. the entropy when θ is the entropy form of Gibbs' relation).*

Assumptions 2 and 3 imply that the dynamical behaviour of the system can be modeled on $\mathcal{L}$ as follows:

$$\left.\frac{dx}{dt}\right|_{\mathcal{L}} = [\mathcal{Q}(x,p)]_{\mathcal{L}} \tag{21}$$

where the i^{th}element of the vector $\mathcal{Q}$ is the net flowrate of the extensive quantity corresponding to the x_i-coordinate.

As it has been shown in [27] these assumptions imply that :

$$f(x,p) = (\frac{\partial^t \tilde{S}}{\partial x} - p^t)\mathcal{Q}(x,p) \tag{22}$$

is a possible contact Hamiltonian function $f(x,p)$ for the representation of the system dynamics. Indeed by calculating the expression of the dynamics generated by the contact Hamiltonian function (22) by using (11) one can find again the expression (21)[1] :

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial f}{\partial p} \\ &= \mathcal{Q}(x,p) - \frac{\partial \mathcal{Q}}{\partial p}\left(\frac{\partial F}{\partial x} - p\right) \\ \left.\frac{dx}{dt}\right|_{\mathcal{L}} &= \mathcal{Q}(x, \frac{\partial \tilde{S}}{\partial x}|x \in \mathcal{L}) \end{aligned}$$

The expression (22) can be inserted in Corollary 1 to obtain the equilibrium points of the system. The following relation is obtained :

$$\left.\left(\begin{array}{c} \frac{\partial^2 \tilde{S}}{\partial x^2}\mathcal{Q}(x,p) - \frac{\partial \mathcal{Q}}{\partial x}(\frac{\partial \tilde{S}}{\partial x} - p) \\ -\mathcal{Q}(x,p) + \frac{\partial \mathcal{Q}}{\partial p}(\frac{\partial \tilde{S}}{\partial x} - p) \end{array}\right)\right|_{(\bar{x}_0,\bar{x},\bar{p})} = \left(\begin{array}{c} 0 \\ 0 \end{array}\right)$$

All the equilibrium points with physical meaning are situated on $\mathcal{L}$, i.e. $\left(\left.\frac{\partial \tilde{S}}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})} - \bar{p}\right) = 0$. As a consequence, the equilibrium condition reduces to $\mathcal{Q}(\bar{x},\bar{p}) = 0$. In other words, the net flowrates of the extensive quantities are equal to zero at equilibrium, which is in accordance with the physical intuition.

[1]Relation (22) has been used in Example 1 for obtaining $f_1\,(x_0, x, p)$

Let us now check the sufficient stability condition given by Corollary 2 for the contact Hamiltonian function (22). A simple calculus gives the following expressions for the partial derivatives of f :

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial p^2}\right)_{i,j} &= \sum_{k=1}^{\gamma}\left(\left(\frac{\partial \tilde{S}}{\partial x_k}-p_k\right)\frac{\partial^2 \mathcal{Q}_k}{\partial p_i \partial p_j}\right)-\frac{\partial \mathcal{Q}_i}{\partial p_j}-\frac{\partial \mathcal{Q}_j}{\partial p_i}\\ \left(\frac{\partial^2 f}{\partial x \partial p}\right)_{i,j} &= \sum_{k=1}^{\gamma}\left(\frac{\partial^2 \tilde{S}}{\partial x_i \partial x_k}\frac{\partial \mathcal{Q}_k}{\partial p_j}\right)-\frac{\partial \mathcal{Q}_j}{\partial x_i}+\sum_{k=1}^{\gamma}\left(\left(\frac{\partial \tilde{S}}{\partial x_k}-p_k\right)\frac{\partial^2 \mathcal{Q}_k}{\partial p_j \partial x_i}\right)\end{aligned}$$

From (20), the restriction of the above expressions to $\mathcal{L}$ is given as follows:

$$\begin{aligned}\left.\left(\frac{\partial^2 f}{\partial p^2}\right)\right|_{\mathcal{L}} &= -\left(\frac{\partial \mathcal{Q}}{\partial p}+\frac{\partial^t \mathcal{Q}}{\partial p}\right)\\ \left.\left(\frac{\partial^2 f}{\partial x \partial p}\right)\right|_{\mathcal{L}} &= \frac{\partial^2 \tilde{S}}{\partial x^2}\left.\frac{\partial \mathcal{Q}}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})}-\left.\frac{\partial^t \mathcal{Q}}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})}\end{aligned}$$

Since the equilibrium points of physical interest lie on $\mathcal{L}$, the expression of the matrix $\Lambda(\bar{x},\bar{p})$ given by (18) can be rewritten as follows :

$$\Lambda(\bar{x},\bar{p})=\left.\frac{\partial \mathcal{Q}}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})}\left.\frac{\partial^2 \tilde{S}}{\partial x^2}\right|_{(\bar{x}_0,\bar{x},\bar{p})}+\left.\frac{\partial \mathcal{Q}}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})} \tag{23}$$

Example 1 (continued). *Assumptions 1 to 3 are fulfilled by the system of Example 1 with* $\mathcal{Q}(x,p)=\kappa\left(T_{env}-p^{-1}\right)$. *It is easy to check that applying (23) gives the same result as in (19).*

The matrix $\Lambda(\bar{x},\bar{p})$ as given in equation (23) is composed of two parts. The first term is linked to the dependence of the flowrates on the intensive quantities while the second term is linked to the dependence of the flowrates on the extensive quantities. In the first term one may also note the presence of the Hessian of the entropy function $\tilde{S}(x)$. As we have considered the entropy form of Gibbs' relation, this is equivalent to the Hessian of the entropy function $\tilde{S}(x)$, which is a symmetric definite negative matrix, according to the second principle of thermodynamics [32]. Although nothing can be said in the general case about the spectrum of $\Lambda(\bar{x},\bar{p})$, let us now consider some particular cases of phenomena that result in a dynamical behaviour of the system, namely the conduction, the convection and the chemical reaction.

- Conduction is a phenomenon due to a spatial gradient in the intensive variables of the system. There is no dependence on the extensive variables. Therefore the flowrates due to conduction solely intervene in the first term of the matrix $\Lambda(\bar{x},\bar{p})$. If the two matrices of the first term are symmetric and if their product commutes, then the eigenvalues of $\Lambda(\bar{x},\bar{p})$ are strictly negative if $\left.\frac{\partial \mathcal{Q}}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})}$ has strictly positive eigenvalues [35]. Let us consider the simple case where the conduction flowrate is proportional to the difference of the intensive variable of the system with the environment: $\mathcal{Q}^{conv}\left(p\right) = L\left(p\right)\left(p - p_{env}\right)$. $L\left(p\right)$ is a symmetric matrix of adequate size and p_{env} refers to the environment. If $L\left(p\right)$ has constant coefficients, the matrix $\Lambda\left(\bar{x},\bar{p}\right)$ becomes :

$$\Lambda\left(\bar{x},\bar{p}\right) = L\left.\frac{\partial^2 \tilde{S}}{\partial x^2}\right|_{(\bar{x}_0,\bar{x},\bar{p})}$$

L and $\frac{\partial^2 \tilde{S}}{\partial x^2}$ are symmetric operators and $\frac{\partial^2 \tilde{S}}{\partial x^2}$ is negative definite. If in addition the product of these two matrices commutes, then $\Lambda\left(\bar{x},\bar{p}\right)$ has eigenvalues with negative real part if L is positive definite and the equilibrium point is asymptotically stable. This is for example the case when L is diagonal, i.e. a spatial gradient in an intensive variable p_i induces a variation only for its conjugated extensive variable x_i. The heat conduction following Fourier's law is an example where L is diagonal whereas the Soret effect or the Dufour effect (matter flows due to temperature gradients and conversely) imply non-diagonal terms in L[2] [36].

- The convection is due to a motion of matter. In the case of forced convection, the net flowrate usually depends on the extensive variables. Therefore the flowrates due to convection solely intervene in the second term of the matrix $\Lambda\left(\bar{x},\bar{p}\right)$.
- The chemical reaction rate depends as well on the extensive variables as on the intensive quantities. Therefore the chemical reaction intervenes in both terms of the matrix $\Lambda\left(\bar{x},\bar{p}\right)$.

The vector of flowrates $\mathcal{Q}$ can be written as a sum of several vectors Q^j of the same length and each representing a different phenomenon (e.g. convection, conduction, chemical reaction) :

$$\mathcal{Q}\left(x,p\right) = \sum_j Q^j\left(x,p\right)$$

The matrix $\Lambda\left(\bar{x},\bar{p}\right)$ can then be written as follows :

$$\Lambda\left(\bar{x},\bar{p}\right) = \sum_j \left[\left.\frac{\partial Q^j}{\partial p}\right|_{(\bar{x}_0,\bar{x},\bar{p})} \left.\frac{\partial^2 \tilde{S}}{\partial x^2}\right|_{(\bar{x}_0,\bar{x},\bar{p})} + \left.\frac{\partial Q^j}{\partial x}\right|_{(\bar{x}_0,\bar{x},\bar{p})}\right] = \sum_i \Lambda^j\left(\bar{x},\bar{p}\right)$$

with a matrix $\Lambda^j\left(\bar{x},\bar{p}\right)$ for each vector Q^j. It is therefore possible to analyze each phenomenon that generates a flowrate and to determine how it contributes to the stability of the system equilibrium.

[2] Still these effects are most often considered as marginal in practical applications.

V. CONCLUSION

The dynamical behaviour of thermodynamic systems can be modeled using the contact structure formalism. More precisely it may be expressed in terms of contact geometry as a contact vector field which leaves invariant a particular Legendre submanifold, representing the thermodynamic properties of the system, and generated by a contact Hamiltonian function representing the phenomena giving rise to the dynamics. From its very beginning, this geometric theory is also strongly related to the thermodynamic principles. The aim of this work was to express the properties of the dynamical behaviour of the system in terms of the elements of the contact structure formalism.

In a first instance, we have discussed the definition of the contact Hamiltonian functions and have given a family of equivalent contact Hamiltonians in the sense that they generate vector fields whose restriction on the Legendre submanifold are identical. In terms of physical modeling they are undistinguishable. In a second instance, we studied the equilibrium points of the system in terms of the contact Hamiltonian function. Furthermore we have shown that for a particular class of contact Hamiltonian functions corresponding to the most common modeling assumptions, these points are their critical points. Using then Lyapunov's first stability theorem, we have established a criterion for the asymptotic stability of the equilibrium points. This criterion depends on the thermodynamic model of the system and the contact Hamiltonian function. Finally we related this stability criterion to the underlying physical phenomena and in particular to the concavity of the entropy function. The stability criterion that we have established can be used to determine if a flux of some extensive quantity entering or leaving the system tends to stabilize or to destabilize an equilibrium.

In this paper we did not consider controlled systems in the sense that the external variables defining the thermodynamical state of the environment of the system has been taken constant. However one may also consider controlled contact systems by expressing the contact Hamiltonian function as functions of some external time-varying variable [14][25][27][37]. Such systems encompass port-Hamiltonian systems, hence one may hope to extend the interconnection and damping assignment passivity-based control (IDA-PBC) methods [5][6] to controlled contact systems. A possible option might be to use this property for the stabilization of unstable equilibrium points by shaping the flowrates of the extensive quantities in order to compensate the fluxes that tend to destabilize the desired equilibrium point.

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